DOCUMENT RESUME

ED 348 229 SE 052 920

AUTHOR Confrey, Jere; And Others
TITLE The Use of Contextual Problems and Multi-Representational Software To Teach the Concept of Functions. Final Project Report.
INSTITUTION Cornell Univ., Ithaca, N.Y. Dept. of Education.
SPONS AGENCY Apple Computer, Inc., Cupertino, CA.; National Science Foundation, Washington, D.C.
PUB DATE 91
NOTE 109p.
PUB TYPE Reports - Descriptive (141) -- Reports - Research/Technical (143)

EDRS PRICE MF01/PC05 Plus Postage.
DESCRIPTORS *Computer Assisted Instruction; *Computer Software Evaluation; Context Effect; Functions (Mathematics); Mathematics Education; Mathematics Instruction; *Problem Solving; *Program Descriptions; Secondary Education; Secondary School Mathematics; *Small Group Instruction; Symbols (Mathematics); Tables (Data)
IDENTIFIERS *Function Probe; Graphing (Mathematics); Prototypes; *Representations (Mathematics)

ABSTRACT

The "Curriculum and Evaluation Standards for School Mathematics (1989)" calls for the revision of existing secondary mathematics curricula which include an emphasis on contextual problems, multiple representations, and the use of computers. The focus of this revision significantly involves the acknowledgment of the key role of the concept of function as an organizing concept around which other important mathematical ideas revolve. This report describes a 2-year project centered around the issue of teaching function concepts utilizing a context-based curriculum in a technology-rich secondary mathematics classroom. The goal of the project was the production, through applied research, of an intermediate-range vision of what mathematics instruction in schools might be like if classrooms were provided with adequate technological resources and appropriate teacher development. The report addresses the following topics: (1) the rationale and purpose of the project; (2) an overall theoretical approach to functions, teaching, learning, and small-group interactions; (3) the design principles, interaction processes, and pedagogical impact of the multi-representational software tool called Function Probe; (4) the use of prototypes within contextual problem settings; (5) the particulars within the implementation process of this project; (6) data collection techniques and research methodology; and (7) research results for the teachers, the students, and small groups of problem solvers. A final chapter offers conclusions about, and implications of, the role of technology in teaching mathematics. An appendix describes the software features, requirements, and availability of the Function Probe tool. (58 references) (Author/JJK)

TM

Final Project Report

THE USE OF CONTEXTUAL PROBLEMS AND MULTI-REPRESENTATIONAL SOFTWARE TO TEACH THE CONCEPT OF FUNCTIONS

Project Director:
Jere Confrey

Prepared by:
Jere Confrey
Erick Smith
Susan Piliero
Jan Rizzuti

CORNELL UNIVERSITY

The Use of Contextual Problems and Multi-representational Software to Teach the Concept of Functions

Final Project Report

Project Director:
Jere Confrey

Project Staff:
Erick Smith, Research Associate
Forrest Carroll, Head Programmer
Simone Cato, Programmer
Paul Davis, Programmer
Jan Rizzuti, Researcher
Susan Piliero, Researcher
Department of Education, Cornell University, Ithaca, New York 14853

Project Teacher: Paula Fistick
Columbus Public Schools

Final Report by:
Jere Confrey, Erick Smith, Susan Piliero, and Jan Rizzuti

This research was funded under grants from the Apple Classrooms of Tomorrow℠ and the External Research Division of the Apple Computer Corporation and from the National Science Foundation (MDR 8652160).

We would like to express our appreciation to the staff and students of West High School, Columbus, Ohio for their participation in this project.

1991

CONTENTS

I. INTRODUCTION

The Curriculum and Evaluation Standards for School Mathematics (NCTM,1989) presents a call for a significant revision of the existing mathematics curriculum. This revision includes an emphasis on contextual problems, multiple representations, and the use of computers. Furthermore, the Standards recognizes the key role of the concept of function as an organizing concept around which other important mathematical ideas revolve. This report describes a two-year project centered around the issue of teaching function concepts using a context-based curriculum in a technology-rich secondary mathematics classroom.

The introduction of new computer technologies into classrooms requires a particularly careful analysis. Enthusiastic supporters predict that the introduction of computers into mathematics classrooms provides the opportunity for radical change and can result in dramatic improvement in mathematics instruction (Kaput, 1988). However, the few studies which do exist suggest that computers are subject to the same dampening effects over time as other innovations. Learning to use computers effectively seems to demand a substantial reconceptualization of the mathematics classroom in content, in method, in forms of interactions, and in assessment (Lampert, 1988; Mandinach, 1989; Wiske, Zodhiates, Wilson, Gordon, Krensky, Lord, Watt, and Williams, 1988).

Yet many of us are convinced that appropriate technical tools hitched to a competent curriculum could offer a genuine alternative to existing practices for teaching mathematics, although this will not simply happen by placing the computers in the hands of technologically capable teachers and students. Only through careful study, documentation, description, and analysis, can we hope to find ways to bring about the reform we envision.

In our previous work, we had begun the development of a computer software program which was designed in response to student methods, was familiar-looking enough to invite entry to practicing teachers, and had the potential to create intentional and dramatic change in the character of inquiry. In addition, we were developing a context-based curriculum which we believed could be used fruitfully with the software to create the potential for substantial change in mathematics instruction. In this project, we were able to complete two units of these materials and work with a mathematics teacher as she used them in a secondary precalculus class. We studied what happened by documenting, examining, and evaluating the possibilities and difficulties which evolved. The goal of the project has been to produce applied research products that are: 1) designed in relation to actual classroom practice; 2) informed by the current research on students' schemes and conceptions; and 3) undergo revision and refinement as a result of examining its use in daily practice in technologically rich classrooms. We sought to create an intermediate-range vision (over the course of the next five years) of what instruction in schools might be like if classrooms are provided with adequate technological resources and appropriate teacher development.

This report is divided into eight sections. Section II presents our overall theoretical approach to functions, teaching, learning, and small-group interactions. The next two sections, III and IV, discuss the development of the multirepresentational software tool, Function Probe©, and a curriculum appropriate for the study of functions at the secondary level. Section V describes the implementation of software and curriculum in a secondary precalculus classroom. In sections VI and VII we describe the research undertaken to evaluate this implementation and finally, in section VIII, we present our summary and conclusions.

II. DEVELOPING A THEORETICAL APPROACH

Concepts of Function

Our research focused on the development of the concept of function. We view a function as a relation between two varying quantities. While we recognize that there is a formal requirement to distinguish a function from a relation by requiring that for each x, there is exactly one y[1], we were concerned more with how students would develop a sense of function in terms of its patterns, its rate of change, its field of applicability, and its representational forms. We were constantly struck by how formal definitions often hide as much as they convey. As with so many mathematical definitions, the simplicity and elegance of a formal definition obscures much of the genesis and richness of the concept. The impoverishment of mathematical ideas when confined to formal definitions and procedures was stated forcefully by Lakatos:

> [In deductivist style,] mathematics is presented as an ever-increasing set of eternal, immutable truths. Counterexamples, refutations, criticism cannot possibly enter. An authoritarian air is secured for the subject by beginning with the disguised monster-barring and proof-generated definitions and with the fully-fledged theorem, and by suppressing the primitive conjecture, the refutations, and the criticism of the proof. Deductivist style hides the struggle; hides the adventure.
>
> (1976, p 142)

A demonstration of how misleading it can be to assume that knowing a definition of function implies understanding the commonly accepted distinctions was provided by Vinner (1983). In his study of 10th and 11th grade students in Jerusalem, 88% of the students (n=129) could state the definition of a function, but, of those students, only 34% (n=43) were able to correctly distinguish functional from

---

[1] However, we see this restriction as a mathematical artifact, important both to avoid the complications of slopes whose denominators are zero and perhaps in part to a preference for certainty in prediction; however, we consider these potentially to be artifacts of representational systems heavily reliant on algebraic symbolism, and perhaps unnecessarily restrictive in the new domains of representation.

nonfunctional relationships in the four specific examples he used. He felt this suggests that the concept images we construct are more influential in guiding our intellectual development than are formal definitions.

According to Vinner, a person's mental picture of a concept is the set of all pictures that have ever been associated with the concept in the person's mind, including any visual representations. Properties associated with the concept may complement these mental pictures. The combination of these things he calls the "concept image."

One question we might ask then is how ought one approach teaching the concept of function in a way that promotes the development of rich and useful concept images and, at the same time, is compatible with standard mathematical definitions? A variety of possible answers were generated by the mathematicians involved in the new math who gave us a set of representations from which we might select: the mapping set diagrams, the use of tables, the use of stretchers and shrinkers on two parallel number lines, the function machines, the use of the box notation for a missing value in an expression and so forth (van Barneveld and Krabbendam, 1982). This variety of notational forms has lost its diversity in the "back to basics" movement, as, typically, one form or another guides the introduction to functions in modern textbooks.

In addition, we might look to the research on functions. Researchers in this area have reported that students believe that:

1. functions should be given by one rule. If multiple rule functions are given, then we have multiple functions. Thus, if an arbitrary mapping is produced, then we have that many individual functions;

2. a function exists only if a legitimate symbol has been created to describe it;

3. a graph of a function should be well-behaved (symmetrical, regular, always increasing and so forth);

4. there should be an operation on x that yields y, thus ruling out nonoperational relations;

5. constant, piecewise, and discrete sets of points are not legitimate functions;

6. specification of a domain and range are not part of the definition of function;

7. there is a directionality in functions from preimage to image and reversing this would cause them difficulty;

8. moving from algebra to graph and visa versa was difficult.

(Vinner, 1983; Markovitz, Eylon, and Bruckheimer, 1983; Herscovics, 1982)

This research demonstrates convincingly that the formal definition provides an insufficient basis for understanding the practices of students. What it does less well, however, is to direct our attention toward an alternative approach that might alleviate some of these difficulties. We believe that the most important place to look for such an approach is in the responses students have provided when working with functions. If we interpret their tendency to view functions as single rules to be a result of legitimizing those situations with which they have had meaningful experiences, this suggests that we need both to create more situations that allow students to build on previous experience and to explore more deeply the understandings and experiences students already have as they begin their study of functions.

Helping students create more viable conceptions of functions demands a detailed analysis of the development of the concept. This type of analysis has been undertaken for linear functions by Schoenfeld, Smith, and Arcavi (1989) and for exponential functions by Confrey (1990a, 1991a). Schoenfeld et al, after analyzing the work of one student while using his software program, Grapher, concludes that the development of a two-point schema for constructing an equation for a line can be a slow and circuitous task. In our work on exponential functions, we stress the importance of assisting students in recognizing the operational character of functions prior to the general algebraic form. We have found that students can often describe how y_1 is transformed into y_2 concurrently with x_1's transformation into x_2 more easily than they can describe how x is transformed into y. By allowing students initially to build a function as a coordination of two quantities varying simultaneously, or as the coordination of an arithmetic and geometric sequence, we have found that their eventual construction of algebraic equations, relating x to y, is more intuitive and more strongly grounded. In addition, we believe that: 1) this perspective emphasizes the rate of change of the function, making it a salient characteristic, and 2) it frequently allows a student to let the x value act as a counter or a tally in keeping track of the number of actions carried out on the y variable, a useful concept in many applications especially those dependent on time.

This raises the question of the role of contextual problems and manipulable tools in learning functions. Contextual problems have been elaborately investigated in relation to classroom practice by Lange (1987) and Treffers (1987) while the use of tools and embodiments is currently under investigation by Nemirovsky (1991), Greeno (1988), and Meira (1991). For Treffers, the starting point for "Realistically Oriented Mathematics Education" includes: 1) attention to reinvention, i.e. building connections to intuitive notions; 2) working across levels of concreteness and abstraction; and 3) designing curriculum in an "historic-genetic" rather than hierarchal structure-dominated fashion. What is emerging from this work is the examination of the development of the function concept within contextual applications.

Although this work has provided curricular examples that assist students in integrating mathematics into their everyday experiences, it has been less successful in articulating students' schemes for approaching the different concepts addressed. By schemes for functions, we refer to the structures students evolve that allow them to anticipate, recognize, act, and reflect on the situational characteristics related to specific families of functions. What we seek to do with our use of context is to identify the variety of schemes involved and to design software and curriculum to develop and enhance these schemes. We tie schemes to operations in a Piagetian sense of operations as "internalized actions" and seek to find the basic actions encapsulated in a family of functions.

In our own work, an example occurs with quadratic functions. We propose that by using the ideas of: i) rate of change; ii) symmetry; and iii) dimensionality, one can improve students' and teachers' recognition of situations where quadratic functions may be appropriate. i) For a quadratic function, if x increases by equal increments, the difference in the y values also increases by equal increments, and thus the second difference in the y-values is constant; that is, an identifiable constancy in the change in the change factor of the dependent variable for a constant discrete change in the independent variable. For example, if height vs. time of a falling body is described by a quadratic function, then the second difference, acceleration, is constant. ii) If the variation in the domain includes symmetric values on both sides of the maximum (or minimum) of a quadratic, the function yields equal values at these symmetric points. When an object is tossed up in the air, one senses a symmetric relationship as it reaches its vertex and returns to earth. In maximum-minimum problems such as a given perimeter yielding a maximum area, one finds two extremes both of which yield no area at all, and these create the symmetric roots of the equation. iii) As the product of two linear equations, one can explore the idea that a quadratic equation is created by multiplication of two dimensions which are linearly covarying. (See Confrey, 1991c; Afamasaga-Fuata'i, 1991, for a more complete discussion of this.)

Nemirovsky and colleagues are undertaking a similar approach in their research on students' development of calculus (Nemirovsky, 1991; Nemirovsky and Rubin, 1991). Using the two primitives of rate of change and accumulation, they are working with students using computer sensors to investigate water flow, air flow and the movement of hand-held cars. By identifying and building situations which call for both rate of change and accumulation strategies, they are assisting students in developing their intuitive approaches to slope and to accumulation of area under a curve.

In a related area, Monk (1989) argued for "the need to understand the pathways of development of the concept apart from its formal development in the material" (p.2). To do this, he wrote, "Although concepts are usually described as nouns, as things, their power comes from what they enable us to do, from the actions they are the compression and crystallization of" (p.2). He introduced the term "functional situation" and suggested six families of questions that can be use as indicators of students' understanding of these situations: 1) forward questions where output must be determined for a given input; 2) backward questions where a value in the domain is found which corresponds to a given value in the range; 3) across-time questions where patterns of change are examined; 4) articulation questions where distinctions among derived quantities are checked (for example, velocity can be viewed as a derived quantity from position/time or from acceleration/time); 5) change of context questions; and 6) multirepresentational questions. He sees these forms of questions as a way of describing a student's "psychological concept of function [as] an organized set of action schemes"(p.17).

From this brief review of the functions literature, we make two claims for the study of functions. The first is that we ought to design problems which use context to stress an operational and rate-of-change-oriented view of functions. The second is that research documenting students' failure to perform according to formal definitions needs to be followed by careful analysis of what it is that students do see in their understanding of functional relations. Is there a systematicity to their "concept images?" If given a choice of representations, how will they build their concepts of functions? These questions lead one to view the learning of functions as an interweaving of formal definitions with the construction of "concept images" that are dynamic and viable within the growing experiential world of the student.

Looking at Students' Methods: Failures or Opportunities?

Much of the research on students' conceptions serves the purpose of reiterating our failures to produce the kind of learned students we envision. It has been demonstrated vigorously that when we press only minimally on students' responses what appears to be competence rapidly transforms itself into confusion and uncertainty. Many use this to express their reservations about the effectiveness of existing practices and to document the unintended outcome of an overly product-oriented assessment system. However, another more positive argument can evolve from this tradition of research. It requires one to deemphasize the tradition of documenting failure and, instead, concentrate on what it is that is legitimate, inventive, and provocatively fertile in student methods. That is, building from the Vinner study, we seek to understand how students' concept images are more than simply an accumulation of experience, how they construct a type of structure which is potentially productive and how they use these structures to make sense within their own experiential world.

Research on student conceptions (Confrey, 1990b; Eylon and Linn, 1988; Champagne, Gunstone, and Klopfer, 1983; Osborne and Wittrock, 1983) suggests that students show evidence of a variety of conceptions in relation to their understanding of an idea. When these conceptions are robust (i.e. enduring and functional for the student), " certain constellations of these belief systems show remarkable consistency across ages, abilities, and nationalities." These belief systems are based in explanatory and descriptive frameworks that are held before formal study and "are resistant to change through traditional instruction" (Confrey, 1990b, p.4).

We would suggest that these conceptions (or schemes, as used by Steffe, and Cobb, 1983) have some of the following characteristics: They are: 1) tied to concepts, i.e. terms used within a discipline to point toward important and recurring ideas; 2) useful and worthwhile from the students' perspective; 3) internalized systems of belief; 4) likely to be able to be reflected upon to some degree; 5) resilient, recurring, persistent and, thus, presumed to have a structure; and 6) generalizable and, thus, invariant across some set of events; and 7) necessary for and the result of the cycle of problematic, action, and reflection (Confrey, 1990a).

In light of this, we can distinguish five types of student conceptions that may operate. These are descriptions from the perspective of the observer. To the student, a conception will appear simply more or less operable. Even though s/he may have varying degrees of awareness and of satisfaction with it, the labels described below make sense only in relation to how a more expert observer [2] might see it. For an observer actively modelling student conceptions, one way to think about these modelled conceptions is to compare them with his/her own. After doing so, the observer may judge the extent to which a conception seems:

1. unacceptable or erroneous (conflicts logically and explicitly with the beliefs of the observer, and the observer cannot imagine changing his/her own conceptions to find it acceptable);

2. limited or bounded (fits with the observer's own perspective but deals with only a bounded subset of the observer's scope of applicability);

[2] Expert observer is a deliberately ambiguous term here, for the expertise of the observer may lie in the level and scope of his/her mathematical knowledge, or in his/her insight into students, classrooms, applications, and the like.

3. alternative (requires the observer to decenter and imagine what the viewpoint is like for the student; it requires the observer, to the extent possible, to change his/her frame of reference);

4. metamorphic or developmental (seems appropriate from the perspective of a naive viewer; it may possess a continuity with the observer's view if developmental or a discontinuity if metamorphic);

5. unrecognizable (recognizable by the observer as so foreign to his/her experience that s/he declines to evaluate it).

We suggest that the tendency of researchers and teachers educated in mathematics is to assume too often that students' responses belong in the first category (and to a lesser degree the second category). Underlying our research and our teacher development work is the assumption that if students are genuinely engaged in solving problems, most of the time their responses fall in categories 2-4. When we are not certain, we are likely to assume that category 5 is applicable. Category 1 is reserved for those instances in which we see either a lack of commitment and thus, a fluctuating or flippant response, or an inconsistency that becomes readily apparent to the student when an appropriate context is provided.

A Constructivist View of Teaching

Given the assumption that students can and do create inventive and viable conceptions, we can begin to articulate what we wish to see in a constructivist classroom. Constructivism commits one to teaching students how to create more powerful conceptions. Variations are expected and nurtured, and the student is given primary responsibility for assessing the quality of the conceptions which s/he constructs. The goal of instruction can be stated as:

> An instructor should promote and encourage the construction by each individual of a repertoire of powerful mathematical conceptions for posing, constructing, exploring, solving and justifying mathematical problems and concepts and should seek to develop in those students their capacity to reflect on and evaluate the quality of those conceptions.

From this goal we make three assumptions:

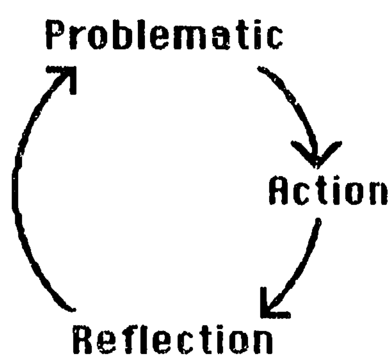
1. Teachers should build models of students' understanding of mathematics. To do this, teachers need to create as many and as varied ways of gathering evidence for judging the strength of a students' constructions as possible. This will allow teachers to create a "case study" for each student.

2. Instruction is inherently interactive; through their interactions with students and their knowledge of their subject matter, teachers construct a tentative path which may allow the student to construct conceptions which are, from the perspective of the teacher, more powerful. However, a teacher must recognize that a student's conceptions will not be identical to his/her own. Success for teacher and student occurs when they are able to agree to agree upon socially shared expressions of their conceptions that are mutually acceptable in satisfying the goals for the course.

3. Ultimately, the student must decide on the adequacy of his/her constructions.

A Constructivist View of Learning.

Confrey (1990a) offers a constructivist model of mathematical problem solving which parallels in many ways the parts of a Piagetian scheme. In modeling mathematical activity as a (creative) cycle, several points are emphasized:



1) For the active problem-solver the problem that appears on a piece of paper is not the same as the "problematic." "A problematic is defined only in relation to the solver and only becomes a problematic to the extent to which and in the manner in which it feels problematic to the solver. When defined this way, as a roadblock to where a student wants to be, the problematic is not given an independent status. The problematic acts as a perturbance, i.e. a call to action." (1990a, p.12)

2) In acting, we "use tools and previously familiar schemes of representation." Although the actions taken by an individual may appear in the form of

socially accepted forms of representation, the actions must take place within the framework of existing schema available to the solver.

3) Reflection involves "monitoring the results of our actions to see if the problematic has been resolved and equilibrium restored." The successful resolution of a problematic may allow us to construct new schema and allow us to "set apart that action or operation (that resolved the problematic) by various processes of naming it, discussing it, objectifying it, or creating it into a tool or representation for further action." In effect, the action of reflection prepares us to take on further, possibly more difficult, problematics. (1990a, p.12)

Thus "knowledge is not the accumulation of information; it is the construction of cognitive structures that are enabling, generative and proven successful in problem-solving." (1990a, p.14).

We used this model in our research to focus on how students approach functions. It influenced our design of curriculum, of the software tool, of forms of assessment, and of our research with students and teachers.

Collaborative Small-group Learning

In undertaking research on small-group work in the classroom, we were interested in developing a basis for answering a larger question: How, from a constructivist perspective, can we account for how learning occurs through interactions with others? Although constructivist researchers have traditionally acknowledged the importance of social and cultural influences on cognition, the emphasis on the use of the teaching experiment (Steffe, 1990, Confrey, 1990a) with the individual child is the predominant methodological approach in constructivist research. Due to this emphasis, constructivist research has been perceived as *only* modelling cognition independently of social interactions. Jerome Bruner, for instance, has characterized the Piagetian/constructivist learner as the "lone child struggl[ing] single-handed to strike some equilibrium between assimilating the world to himself or himself to the world" (1985, p.25).

It is our position that there is no fundamental contradiction between constructivism and issues of social cognition, but that the methodological emphasis in constructivist research has been concentrated on the individual learner. A strength of constructivism is that it allows for and values individual diversity. This does not imply, however, that constructivism must necessarily conflict with concepts of social cognition. Constructivist researchers do recognize the interactional, cultural, social, and political quality of the teacher-student exchange and of student-student exchange (Cobb and Steffe, 1983; Confrey, 1990a). In this work we begin to develop these connections more explicitly.

The work of Vygotsky and others who have worked in the sociohistoric tradition provides a potentially fruitful area for this work, developing from a distinctly alternate approach to cognition. The issues and concepts arising from the sociohistoric approach, looked at from a constructivist perspective, help to provide a cognitive/epistemological approach to mathematics and the learning of mathematics that is not necessarily in conflict with either perspective. The intent of this work was to develop a basic theoretical model for understanding the social construction of knowledge from a constructivist perspective and to use that model as a lens to focus on small-group interactions in the classroom. However, in addition, this work provides foundational support for further research in this area.

Activity, Problematic, and Negotiated Consensus

At its inception, this research focused on the issue of the problematic as it evolves for students working in a small-group setting. Based on the problematic-action-reflection model of Confrey (described above), it was expected that the actions taken by an individual student within a group setting would depend both upon the student's problematic and the extent that the group's actions were seen as potentially resolving that problematic. Through communicative acts, a consensus should be negotiated that would allow the group to present a mutually acceptable solution in the sense that the solution viewed as a representation would be seen by each individual student as potentially resolving their problematic. This approach would also provide evidence for what Roschelle and Behrend have called "collaborative learning":

> . . . a coordinated, synchronous activity that is the result of a continued attempt to construct and maintain a shared conception of a problem.
>
> (1990, p.1)

Although it may be argued that the process just described must take place at some level in order for a group to work toward a problem solution, it is another issue to create a situation where this process can be documented. Given a goal of creating a collaborative learning situation, how do we get students to see the task that they are about as a process of negotiation? How do we get students who have had eleven years of conventional mathematics instruction to view mathematics as a subject open to individual interpretation? In short, how do we get students to value diversity in a mathematics classroom? Some of our successes and failures in this attempt will be described in the section on results. However, in trying to answer these questions, it has become clear that a larger framework was needed, something akin to Leontyev's concept of "activity" which he describes as "specific processes which realize some vital, i.e. active, relation of the subject to reality . . ." (1981, p.36). It is the most inclusive unit of analysis of the mind and includes the "goals, means, and constraints operating on the subject." (Cole, 1985, p.151). This concept will allow us to discuss the setting in which individuals develop problematics and groups develop a shared concept for a problem. Thus three concepts form the basis for the analysis:

1. Activity is the social frame in which individuals take actions. In accordance with Leontyev, activity is essentially connected to social setting, but it should be emphasized that from an educational point of view the individual's construction of the setting is important. From a social perspective, the activity for students participating in the class could be defined by the curriculum and other factors involving the school setting. However, in order to understand how individuals are operating within the mathematics class, it will be equally important to try to understand how students create their individual constructions of the social activity. The primary question that determines activity is: What does the individual student see herself to be about in taking the class?

2. Problematic: When looked at in the context of peer collaboration, it is particularly important to keep in mind the social constraints which act upon the individual problematic. Although each student comes to a problem situation with a unique perspective, each perspective has been constructed within the forms of cultural tools. Thus, it would be a mistake

to assume that the individual constructive process, what might be considered 'finding the problem within the problem,' removes the problematic from the constraints of the social world. Language in its role in intrapsychological thought forms a significant component of the building blocks of the problematic, and of communicative acts undertaken. As von Glasersfeld states, "Every individual's abstraction of experiential items is constrained (and thus guided) by social interactions and the need of collaboration and communication with other members of the group . . ." (1990, p.26). We should not assume that a problematic remains static throughout the problem-solving process. Through interactions with others, actions taken by others, and cognitive conflict, problematics evolve. What is important is that the individual not 'give up' her problematic and turn her authority over to others for what constitutes a solution.

3. Negotiated consensus: Actions or statements produced by the group for which they "agree to agree" (Confrey, 1990a). Negotiated consensus arises through both cognitive conflict and mutual appropriation. The primary difference between negotiated consensus and "taken-as-shared" knowledge as used by Yackel, et al (1990) is in emphasis. Yackel et al tend to use taken-as-shared knowledge as implicit and general agreements within the group that develop and allow for effective communication among group members. We will use negotiated consensus more for explicit and specific situations, such as proposed solutions to a problem or an aspect of a problem for which we believe we can cite evidence that relates the modelled problematic of individuals to the consensus negotiated.

Although we assume that as we probe more deeply into small-group interactions we will require a more extensive model with further development of theoretical terminology, the three theoretical constructs described above should be basic to such a model. If we can develop models of individual students that allow us to see them involved in an activity within which they develop a problematic in relation to a specific problem, and then understand how the student participates in negotiation with others to derive a meaningful solution, we will have taken a large step toward developing our understanding of collaborative learning.

In this section, we have presented a description of the overall framework within which this research was conducted. This is important to specify, because the change we sought to encourage in the ACOT classroom was intentionally tied to this vision of educational activity. We sought to design software that would support a constructivist view of teaching and then to work together with a teacher and students in implementing the software with curricular examples in an Apple Classroom of Tomorrow.

III. A MULTIREPRESENTATIONAL SOFTWARE TOOL

Software and Representational Issues.

Exploring student conceptions and historic perspectives encourages one to question the adequacy of the existing forms of representation. It further requires a skepticism about the "transparency" of any representational form. Finally, it demands that one work toward providing students an environment rich in multiple representations with which to express their ideas.[3]

Representations are vehicles for communication -- both for self talk and for interpersonal talk. As they are used for a variety of purposes within the context of different problems, the meaning of the representation changes. For example, in our teaching experiments, we have found that for students, viewing a number line as a location of points for representing events over time (i.e. as a timeline) is quite different from viewing a number line as a means to express the comparative size of a set of minute objects. Thus, one must recognize that, even with a traditional form of representation, students build their understanding of that representation as a vehicle of expression.

In effect, this argues that representations and ideas are inseparably intertwined. Ideas are always represented, and it is through the interweaving of our actions and representations that we construct mathematical meaning. We consider symbol systems, formal proofs, calculator keystroke records, graphs, figures, statements of contextual problems, and spoken language as equally illustrative of representations. As stated by Goldenberg,

---

[3] "Transparency " is a term used in the literature to indicate, in a sense, a match between concept and representation. A representation which perfectly matches a concept is transparent, that is, the representation reflects an undistorted view of the concept. This notion depends on the idea that a concept exists independently of our representations. However, thinking of Vinner's concept image, one would think of a concept, for an individual, as composed of all the representations which one associates with that concept. Within this framework, the idea of a transparent representation is meaningless because concept and representation are inseparable. This leads quite naturally to the idea that concepts become richer and more viable when they are investigated through multiple representations. It is in this sense that we reject the possibility of a "transparent" representation and emphasize, instead, the importance of the coordination of multiple representations.

> . . . each well-chosen representation views a function from a particular perspective that captures some aspect of the function well, but leaves another less clear: taken together, multiple representations should improve the fidelity of the whole message.
>
> (1988, p.7)

In accordance with other researchers (Roseberry and Rubin, 1989; Pea, 1987; Goldenberg, 1988; Feurzig, 1988, personal communication), we suggest that devising and researching the use of multiple representational forms is useful for its potential to:

1. highlight different aspects of the concept;
2. lead to a convergence across representations that may improve or strengthen our depth of understanding;
3. promote examination of the potential conflict among forms of representations;
4. allow for assessing how changes in one representation affect another;
5. illustrate how alternate forms of actions in a representation can lead to the development of diverse student schemes;
6. cross the boundaries of traditional content disciplines (algebra, arithmetic, and geometry);
7. provide for situations for students to conduct their own investigations of ideas;
8. provide students opportunities for feedback, revision, and reflection.

Design Principles for Function Probe©

This perspective on representations played an important role in the development of Function Probe (Confrey, 1989), a multirepresentational software tool which combines graphs, tables, algebra, and a calculator to allow students to explore the idea of functions and relations in diverse forms. In addition, Function Probe is based on what we have called "student-centered design" (Confrey and Smith, 1988), that is we look to the actions we see students undertaking while solving problems as a basis for the kinds of possible actions we incorporate into the design of Function Probe. The primary design principles used with it included:

1. Each representation must have its own integrity. We sought to avoid unnecessary dependencies between representations. For example, one should be able to carry out transformations on graphs by taking action directly on the graphs, rather than being required to make the changes algebraically. In addition we tried to use designs that remained relatively true to the action. Thus graph transformations are carried out through a mouse action designed to remind the user of the physical feeling of actually stretching or translating the graph. When stretching a function, an anchor line is displayed which acts as an axis of invariance for the stretch. The anchor line need not be an axis, but can be moved to any position on the graph. After setting the anchor line, one places the cursor over the function, producing a square "unit box with a spring inside." As one drags the mouse, the box and enclosed spring will stretch, and the register in the top section of the window keeps track of the coefficient of stretch.

2. We distinguished between curricular and design decisions. Function Probe is not tied to a particular form or sequence of a curriculum. Thus, for example, we would argue that reflecting as a *graphical action* can be done around any horizontal or vertical line in the plane and that to constrain the action to a reflection around an axis is to constrain design based on traditional curricular approaches to algebra. Thus the function $f(x) = x^2 + 3$ can be reflected around the x-axis algebraically by multiplying by -1 (i.e. $y = -1(x^2 + 3) = -x^2 - 3$). In the traditional curriculum, this may be the only vertical reflection covered due to its straightforward algebraic connection. However, one can imagine the graphical image of many vertical reflections. In particular, one might be interested in reflecting the graph around its vertex (i.e. around the line $y = 3$). Although this is a straightforward graphical action, it requires a higher level of sophistication to understand algebraically that this is represented by the equation $y = -x^2 + 3$. We would claim that the decision whether to include such reflections in a mathematics class is a curricular decision. Including them in Function

Probe was a design decision based on the types of actions appropriate to a graphical representation.

3. It is essential to make careful distinctions between designing a learning tool and designing an expert's tool. Although specific curricular issues were not included in the design of Function Probe, certain pedagogical issues played an integral part in its design. In particular, we drew a line between features that would enhance a student's constructive efforts in exploring and solving problems from those that might lead, restrict, or deny the opportunity for such exploration. For example, it is important to be deliberate about one's choice of automation. One option is to make automation progressive. In design, we deliberately chose not to automate features for which we believed student interaction was essential in building stronger understanding. For instance, Function Probe does not automatically scale a graph, but requires a student to make decisions about scale and the placement of tic marks. We believe that assessing the relevant domain and range for a graphed function is an important aspect of working in a formal or a contextual mathematical situation.

4. Each representation should incorporate features that make available the kinds of actions which students are most likely to use. Examples of this are common in Function Probe. In the table window, we built in a **Fill** command which allows students to fill a column by adding, subtracting, multiplying, or dividing by a constant value. This enables them to construct functions by coordinating two separate fills, called co-fills. The most recent version of the program allows an even broader array of fill commands, as well as ways to link columns to create dependencies without needing formal algebraic equations. We also included a difference command (Δx), to calculate differences between consecutive values, and created a ratio command (and a new symbol, ®x) to give the ratio of consecutive values.

5. The meaning which students construct for the concept of function will vary across representations. The ability to construct these meanings independently should be protected. We knew we had had previous success working with students using a keystroke representation for

functions. When students carry out keystrokes on a calculator as step-by-step procedures, they tend to engage more closely with the operational character of the function. We found a keystroke record to be a wonderfully useful representation in teaching inverses of functions, which can lead to surprising results (Confrey and Smith, 1988). In designing the calculator window for Function Probe, we designed a button-maker by which the student can copy a line of keystroke code, replace the number which varies with a ? and save that as a procedure. Visually it is saved as a calculator button which the student can name. This name and associated procedure are recorded in the history window and the button is added to the display. This is an example of how the keystroke notation combined with an input symbol can become a legitimate alternative form of representing a function.

6. Passing functions between representations is an important way to encourage the interweaving of ideas of functions. Function Probe allows one to define a function in any representation. Functions defined in the graph window are labelled sequentially as g1(x), g2(x), . . .; in the table window, h1(x), h2(x), . . .; in the algebra window, f1(x), f2(x), . . .; and in the calculator, j1(x), j2(x), ... Students can pass functions between representation through either a **Send** or a **Call** procedure.

7. One must balance possible limitations of traditional notation against the risk of introducing newer forms. Traditional calculators have a single y^x key for calculating powers. Such a key, however, makes no distinction between, for example, the two functions, $y=4^x$ and $y=x^4$. Since our emphasis was on concepts of function and variable *and* since we wanted our calculator buttons to have an inverse relative to the functional relation (by holding down the "option" key, the buttons turn into inverses), we included two buttons on our calculator, an a^x button and an x^a button. On these binary keys, a student must first input the x (or variable) value, then press the button, then enter the fixed parameter, a. For example, the sequence 3, a^x, 2 calculates 2^3, whereas the sequence 3, x^a, 2 calculates 3^2. The difference in these buttons is emphasized when the "option" key is

held down, for the a^x key turns into a $\log_a(x)$ key, and the x^a key turns into an $x^{1/a}$ key.

8. The most direct feedback in multiple representations is through convergence or disparity. At the current time, student feedback is given through the convergence or lack of it in the various representations. However, after conversations with Wayne Grant, Apple Computer Inc., we believe that various forms of reflective feedback and help options can be designed to strengthen this aspect without undermining the student's independent activity and autonomy. We hope to engage in some future design work in this area.

9. A history is kept of the use of each representation and it is available to students. Students can benefit by learning to reflect on their actions, thus Function Probe was designed to record a keystroke history which is always available to the students. Summary data of the use of windows, their order and their duration may also be recorded for research purposes.

Cyclic Design Processes, Backtalk, and Their Impact on Our Understanding of Students' Views of Function

It should be stressed that we do not see design as a single-step process, but instead continually cycle through our design. Sometimes this is a process of our own reconceptions of design only, but usually it is mediated by our observations of the use of Function Probe and the influence of those observations on our own understanding of functions. A model for this type of cyclic design, using what we have called "backtalk" from students, programmers, teachers, and other educators, is shown in Figure 1.

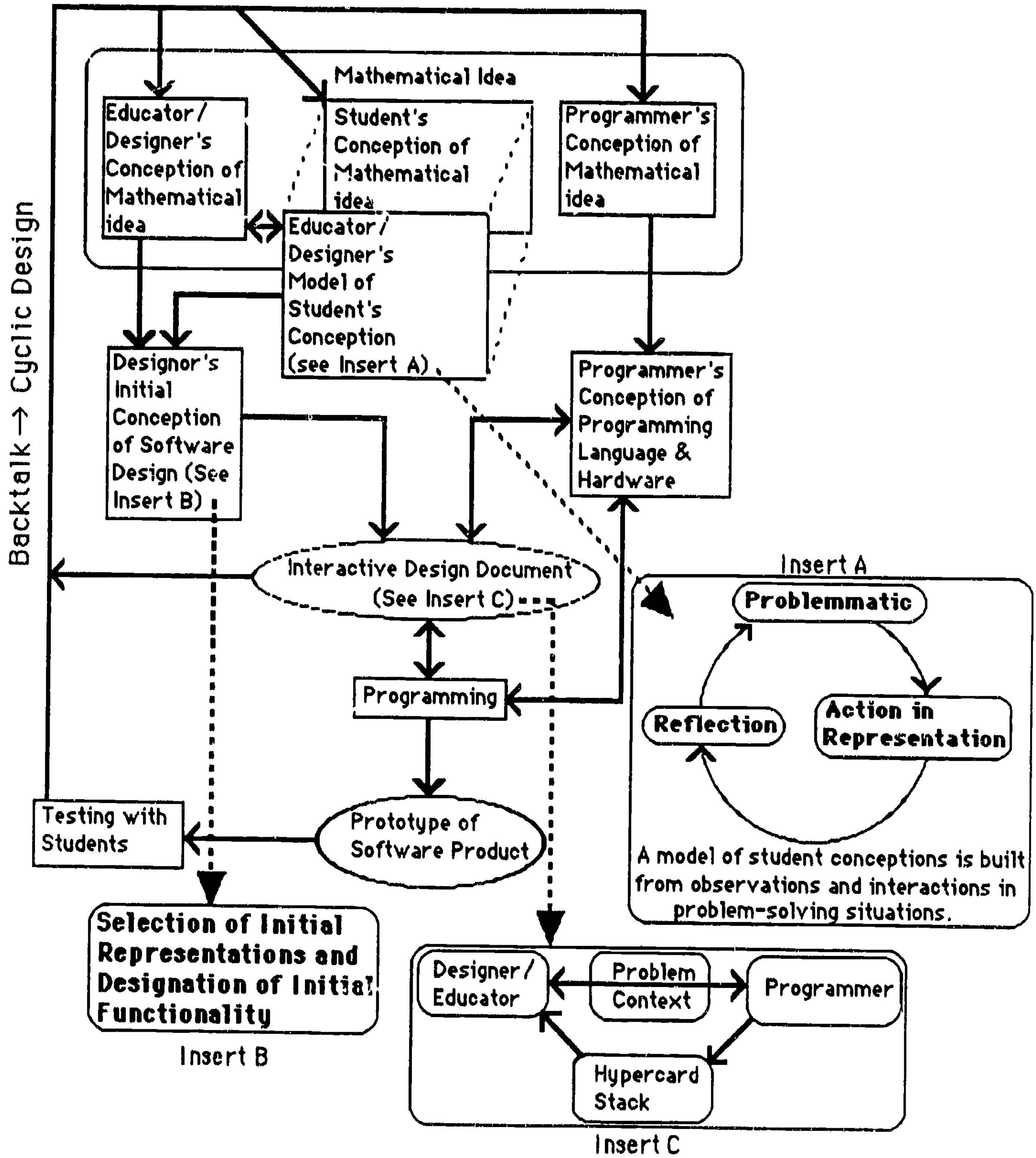


Figure 1

Two examples will illustrate the concept of "backtalk:"

1. A student using the calculator will often work with a linear situation in the following manner. " I start with a certain amount (125) and decrease it by another amount (2.5)." S/he inputs *125- 2.5=*. "Then, I decrease it another

2.5: *125- 2.5- 2.5=*." Then to predict the decrease after 10 turns, the student will write 125- 10*2.5= and reads this as *ten two-point-fives*. After building a button, this becomes: *J1: 125 -* ● *2.5=. One could translate this literally as: *constant value minus variable (or counter) times the constant adder*. Contrasting this to the y=mx + b version (y= -2.5 x + 125), one can see that the button version (y= b + xm) potentially encourages a representation more directly related to the way a student may visualize the action in the problem. We have found that it is easier for many students to see 2.5 as a multiplicand -- that which is repeated - than to see it in the algebraic form where the 2.5 is the multiplier.

2. Watching students and experts use Function Probe, we have increasing evidence that when faced with a contextual situation, some users create a sense of the function to decipher their scales while simultaneously using tentative decisions on scale to assist them in approaching a sketch of a function. Thus, scaling becomes an issue of mutual clarification of a function and the plane used to represent it. Viewing it this way, one sees two actions occurring concurrently. We believe this is a particularly fruitful area for future investigation and are currently building in a sketching tool to make it possible to begin with a sketch of the graph.

As illustrated in Figure 1, the design process started with a model of the students' understanding of the concepts. This model was built from extensive teaching experiments with students in which they were given intriguing problems to work on and provided rich resources with which to work (Confrey, 1989a; 1989b). These interactions between student and interviewer were videotaped and transcribed and analyzed using an approach of seeking the student's problematic (idiographic interpretation of the problem), their choice of action and method of representing and their reflections on the topic (Figure 1, insert A.).

The researcher, the designer and the programmers then all brought their own conceptions of the mathematical idea into the design process. In our project, the designer planned the initial functionality of the representations (Figure 1, insert B) and then the team, which consisted of two staff members (with experience working with pre-calculus students in small-group problem solving sessions and in computer laboratories) and two programmers, developed the design document by working through the mathematical problems on which students had been working. The staff would present a typical student's path through the problem-solving process and discuss alternatives they had observed. The combined educator/programmer staff would then discuss how these student methods might unfold in a computer environment, carrying through each step of the process in detail to the conclusion.

By the next staff meeting, the programmers, working with Hypercard©, would develop a sequenced step-by-step "slide show" of how they would envision this problem-solving process taking place in Function-Probe©. This Hypercard© representation would, as near as possible, give exact duplications of each step as they would appear on the screen to a user of the program (Figure 1, insert C.).

This whole process was then repeated using prototypes of the tool and early versions with students and then feeding the redesign proposals back into the process.

IV. THE USE OF PROTOTYPES AND CONTEXTUAL PROBLEMS

Prototypes.

Working with a concept from Eleanor Rosch (1977), we think there are certain prototypic functions that need to be taught to students in relation to the actions, contexts and representations from which they evolve and in which they are encountered. We organize our materials around the introduction of these prototypes through the use of contextual problems to help students see how actions used to build the prototypes can be associated with actions through which they understand a problem situation. We then work with students to understand how they can transform these prototypes to fit specific situations they have encountered. Each function has its own character. As an example, an extensive discussion of the concept of exponentiation can be found in Confrey, 1990a; 1991. The basic prototypes we include in our precalculus work include: $y=x$; $y=x^2$; $y=x^3$; $y=1/x$; $y=\sin x$; $y=\cos x$; $y=a^x$; $y=\log_a x$; $y=[x]$; and $y=|x|$. We believe that to develop the power to use functions within their diverse fields, students should learn to create schemes for recognizing circumstances that are likely to be well modelled by these functions and to adapt, combine, or compose them as needed.

The Use of Contextual Problems.

Students typically see secondary mathematics as divorced from practical application. Their secondary mathematics experiences are dominated by the algebraic and symbolic manipulations. Applications, when they are offered, are provided "after the fact," as the application of concepts and skills which are hopefully already mastered. In our curricular materials we take a contrasting view. We believe that problems can be written that "unfreeze the mathematical idea" by reclaiming its roots in human activity. Thus, we seek to write problems which <u>suspend the concept in a form of human action</u>.

For example, in teaching linear functions, students need to build an understanding of how an initial amount and a constant additive rate of change can be seen as a y-intercept and the slope of a straight line, or as an initial value and an additive counter, or, in a table, as an initial value followed by a **Fill** using a constant addition or subtraction, or the like. If students can come to recognize that these characteristics are necessary for a linear function, they will be successful in solving linear functions in contextual problems. We do not deny that the subsequent coding of these parameters into some form of symbolic notation also requires significant attention, we are only suggesting that the coding part of the activity 1) varies across representations and 2) becomes far more straightforward when the initial expectations for linear functions are met.

In teaching students the concept of a linear function, one of the problems we use is the stepping-stone problem:

> Suppose you were asked by an elderly neighbor, Agnes Wholesum, to lay a stone path from the back door of her house to a bird feeder forty-four feet away in her back yard. She has purchased 15 circular concrete "stones," each 1 foot wide, and would like you to place them equally spaced across the back yard with the last stone touching the feeder. The distance from the house to the first stone is up to you, so you may make it different from the spacing between the stones. Before going to her house to do the work, you have decided to make a plan.

We use it to introduce linear functions, because in comprehending the situation, the student must very quickly deal with the requirement that the stones will be equally spaced. In addition, when they are asked to define a function that maps the sequential number of the stone to the distance of that stone from the house, they must deal with how one defines distance -- from the near edge, the middle, the far edge, etc. Students build a "feeling," the internalized belief that for each additional stone, the distance to the house increases a constant number of feet.

We are not implying that with such a problem there is uniformity in the students' approaches. We have seen students describe the problem in the following ways:

1. There are 44 feet. If there were 45 feet, each stone would occupy three feet (2 feet of space and one foot of stone). Thus, measuring to the <u>far</u> side of the stone, the equation d=n3 makes one foot too many. Thus they slide the *whole* path one foot toward the house giving the equation: d = n3-1, where d is the distance to the far side of the stone.

2. If three feet are allowed for each stone` (two feet of space and one foot for the stone), it is one foot too long. Thus the <u>near</u> side of the first stone is

placed only one foot from the house. Thus the equation is d=1+(n-1)3 or 3n-2 where d is the distance to the near side of the stone.

3. The distance to the far side of the last stone is 44 feet from the house. Since there are 15 stones, you can write d = (44/15)n.

4. The first correspondence is 1,2; so the relationship might be d=2n. But in the second case, (2,5) the result d=2n falls short by 1 producing (2,4). In the third case, (3,8) , the prediction, (3,6) fails short by 2. And in the fourth case (4,11), the prediction which would be (4,8) fails short by 3. Thus the answer is always off by n-1. Thus the answer is d= 2n + (n-1), where again d is the distance to the near side of the stone.

In a sense, describing the students' methods goes beyond the contextual problem, for how they are derived is intimately linked with the use of the software. However, their strategies do assist us in making well-informed guesses about their cognitive activities. In this sense, it can be argued that the last strategy does not yield a sense that the student has constructed a connection between the form of the equation and the constant difference between stones.

We seek to design problems that:

1. invite entry through a felt discrepancy;
2. create a sense of expectation for possible solution;
3. highlight the operational character of the function;
4. allow for the use and entry by familiar methods; and
5. yield for solution by multiple methods.

V. IMPLEMENTATION: INTRODUCTION INTO A SECONDARY CLASSROOM

Site

This project was conducted in the twelfth-grade Apple Classrooms of Tomorrow (ACOT) mathematics classroom at West High School in Columbus, Ohio. The classes were held in a room containing 30 Macintosh SE computers connected on a local area network, Waterloo MacJanet© (Watcom Products, Inc., Waterloo, Ontario).

The Teacher

The teacher, Paula Fistick, was a veteran mathematics teacher in her sixteenth year of teaching mathematics. With the exception of two years in another state, all of her teaching experience has been in the Columbus Public Schools, and the last ten years has been at West High. Three years ago, Paula had gone to her principal, requesting a change from the routine of the regular classroom and indicating an interest in working with a special district program for potential dropouts. Her principal wanted to keep her on his staff and in the math classroom, and proposed that she apply for an opening in the ACOT program. At that time she says she "hated" computers, and considered herself computer phobic. Despite these tenuous beginnings, she now says, "I really didn't know what I was getting into" but "I am just so happy that I've ended up where I am." She cites two reasons for this satisfaction. One is having the opportunity to become familiar with computers, particularly in classroom situations. The other is the collegiality and peer support among her fellow ACOT team members. The ACOT team often describes itself as a family, with the teachers and students comprising a "community of learners," where there is "no threat, no competition" (3/14/89).

Paula describes her teaching prior to ACOT as very traditional, relying primarily on lectures and overheads, with a daily routine consisting of review problems, presentation of objectives, the actual lesson, student examples, and homework (9/13/89). During the 1988-89 school year, the ACOT mathematics teachers, together with the program supervisor, decided to implement an individualized instruction model for mathematics classes. Her experience with individualized instruction convinced her that although students need more teacher guidance and support than a strictly individualized approach provides, she would also not want to go back to a lecture format. Feeling that "I want to keep my time in front of the classroom to a minimum" (3/14/89), Paula envisioned a teaching model for the 1989-90 school year that would incorporate the best of both traditional and individualized instructional models and include ways to increase computer use in class above the previous level of about twenty minutes per week.

The Students.

There were twenty-three seniors in the ACOT precalculus class in the fall semester and twenty-two in the spring semester. The class was evenly divided between male and female; fifteen (68%) were white and seven (32%) were Afro-American. Sixteen of the ACOT seniors had been part of the ACOT program since its inception at West High School in the 1986-87 school year. All had been in the program at least one year prior to this project. Admission to the ACOT program is based on random selection of interested eighth grade students who have scored above the 37th percentile on the California Test of Basic skills with controls for gender and racial balances.

As eighth graders, the average score on tests of basic skills for this particular class was the 75th percentile. In 1990, after seven semesters of high school, they had a grade point average of 2.715 (on a scale of 4.0); two of the twenty two students rank in the top 10% of their graduating class, 16 in the top 40%, and 3 in the bottom 40% of 253 seniors. According to their teachers, many of these students would not have undertaken four years of mathematics instruction if not for the ACOT program.

Duration

The project took place over the period March 1989 to March 1990 and involved four phases:

1. An initial site visit in March, 1989 to meet with ACOT teachers and staff and to initiate plans for the work.

2. A week-long teacher development session at Cornell during July, 1989. This was attended by Paula Fistick, two ACOT colleagues, and the research staff.

3. A linear functions unit used in the precalculus class from September 5 to October 12, 1989.

4. An exponential functions unit used in the classroom from January 23 to March 8, 1990.

Our view of implementation was not one in which one deposits the innovation as a product with the teacher and expects her to enact it faithfully and faultlessly, but one in which we worked actively with the teacher to cooperatively develop ways to implement the materials in her classroom. We recognized that in order to preserve the spirit and intent of the innovation, we were deliberately asking her to change many of her beliefs about mathematics, mathematics learning and teaching, and assessment. We recognized that her implementation would necessarily transform the software and curriculum in ways that we could not anticipate; some of which we would welcome and others of which we would not. We wanted to describe that process of change. We explicitly indicated that we did not know how to use the curriculum effectively in her setting, and that she held the final right and responsibility for decision-making on all matters of instruction. We sought to establish a partnership where each of us could apply our expertise toward making the innovation work. Finally, we reiterated to her that although some of these materials had been used at the college level, her use of them would be pioneering new territory, and we had no predefined standard by which she would be judged. We wanted simply to study how the project evolved, while assisting her.

Problem-based Curricula

During our initial site visit in March, meetings between the ACOT staff and the Cornell research project staff had been held to determine the level and content of the curricula to be used in the project. It was decided that a unit on linear functions should be used to start off the precalculus course, that a unit on exponential functions should be used later in the course, and that curricular problems for these units would be prepared by the Cornell researchers. These two problem-based curricula were developed by the project staff in the summer and fall of 1989. It was also decided that units on quadratic functions and trigonometric functions would be taught using the textbook (*Precalculus Mathematics*, Demana & Waits, 1989) with supplemental problems from Confrey's precalculus course at Cornell and others written by the ACOT teacher.

The linear functions unit contained eight contextual situations and accompanying questions which we called "problem-solving problems." The teacher supplemented these problems with routine exercises from the textbook and with worksheets containing nonroutine, conceptually oriented questions. The exponential functions unit contained six contextual situations as well as problems involving the generation of geometric figures and geometric sequences.

Teacher Development

Since the instructional technique, based on the use of interactive software to solve contextual problems, was quite different from traditional approaches, it was decided that the innovation could not be implemented without a teacher development program. In July 1989, the participating teacher and two colleagues from the ACOT staff (one in mathematics, one in science) came to Cornell for a week to familiarize themselves with Function Probe and the linear functions curriculum and to discuss with the project staff how we could best implement the changes. The teachers assisted in the further development of the curriculum, writing their own contextual problem and working on nonroutine problems to link the contextual problems with the more traditional precalculus curriculum. We continued to work with the participating teacher on these materials throughout the course of the study.

Teacher development activities for the exponential functions unit were initiated over the course of three days in early January, 1990. The teacher worked through the exponential functions unit on Function Probe and planned some of the instructional activities for the unit. Teacher support and development continued through the winter site-work between the teacher and project staff.

VI. RESEARCH METHODOLOGY

The research planned around the implementation of the innovation was to engage in two significant observational periods: five weeks in September and October on linear functions and five weeks in January-February on exponential functions. Two additional observational periods were planned with the teacher. The first was to observe instruction in November for three days on trigonometric functions and the second was to observe her development and implementation of a unit on quadratic functions, undertaken as an independent project with the project director.

The team used the method of triangulation in data gathering. One researcher, Jan Rizzuti, studied the development of individual students as they learned to use multiple representations in relation to functions. Another researcher, Erick Smith, concentrated on the small-group interactions, again looking at the impact of the software and curriculum on students' construction of a negotiated consensus for solving problems. The third researcher, Susan Piliero, studied the teachers' development of pedagogical content knowledge -- her beliefs about mathematics, mathematics teaching and learning and her routines and practices. This person was the only team member who was not available to the teacher as a resource for teacher development. Her role was to work closely with the teacher, discuss her plans and her reflections, and to stay in the role of confidant. We felt that if she were viewed as a development specialist, the teacher would be less likely to speak candidly to her. Dr. Jere Confrey provided the initial design for the curricula, worked with the teacher on teacher development issues, administered the project, supervising the research team's work, and assisted in the analysis of the data.

Data Collection

Data were gathered from the following sources: 1) videotapes of all teacher development and class sessions; 2) audiotapes and videotapes of all interviews (teacher, individual students, groups of students); 3) printed records of student work on Function Probe; 4) pretests and posttests for each of the two curricular units; 5) observation notes taken during classes; 6) copies of teacher-written tests and worksheets and student work on assignments and tests; and 7) trace records of students' interactions using Function Probe.

Teacher

Data was collected during all four phases of the project:

1) During the initial site visit, the research team was able to attend classes taught by the teacher and an interview was conducted with the teacher on current instructional practices.

2) All teacher development sessions were videotaped. Interviews with the teacher were videotaped.

3, 4) Classes during the linear and exponential units were videotaped and field notes recorded. Daily interviews with the teacher were audiotaped, and lesson plans, handouts, quizzes and tests were collected.

In addition, one member of the research group travelled to Columbus during trigonometric and quadratic units taught by the teacher to attend classes and conduct further interviews.

For the exponential unit, tests were prepared for and completed by the teacher both before the development sessions were held and after the development sessions but before the onset of instruction on exponents and logs, and again after the completion of the unit. The tests were then discussed by the teacher and a researcher and all discussions were audiotaped and transcribed.

Several steps were taken to preserve the veracity of analyses and to uphold the ethical standards we had set for the research. First, the teacher was always made aware of the intentions of the research group and the methods to be used for data collection. Second, tapes and field notes were made available for the teacher to access at any time. Third, as themes emerged from the research they were shared with the teacher, and interpretations were negotiated. Progress reports and research-based papers were reviewed with her as they were generated, and her opinions and feelings were solicited. And fourth, it was repeatedly stressed to the teacher that the authority for the classroom remained with her and not the research group, and that she retained the right at all times to make additions, omissions, or changes to the curriculum as she saw fit.

Students

Interviews were held with individual students during students' study halls and lunch periods or after school. All interviews were voluntary, and students were paid five dollars per forty-minute interview. The interviews were composed of a structured part, containing specific questions to be answered, and an unstructured part in which a student worked on a problem-solving problem.

Data on group problem-solving efforts were recorded during class periods in the fall, but, due to recording problems in a noisy classroom, the winter group data were gathered outside of class working with a single group in a quiet room. Group interviews were voluntary and students were paid five dollars per forty-minute interview.

All student work on Function Probe was saved daily and all student assignments were collected and copied. Pretests and posttests were administered for each of the curricular units.

VII. RESEARCH RESULTS

The Teacher

Although analysis of the data is still under way, certain salient themes can be identified and instantiated. The results will be presented in the following order: subject matter expertise; conceptions of mathematics teaching; conceptions of mathematics; student methods; and classroom routines.

Subject-matter Expertise

This section describes the development of the teacher's own mathematical understandings, and describes instances in which these understandings affected her instruction.

a) Knowledge of mathematics

Paula has shown evidence of substantial growth in her own subject matter knowledge. The strongest evidence comes out of the unit on exponential and logarithmic functions, where her subject matter knowledge was initially weakest. Comparisons of written pre- and posttests and follow-up interviews on the substantive content of this unit show considerable development. Her knowledge of exponential functions was restricted primarily to competence in the laws of exponents and the applications of those laws to the algebraic manipulation of exponential expressions and equations, i.e., what is typically included in a traditional second-year algebra curriculum. Prior formal knowledge of logarithms had essentially been forgotten, since she had not had any recent experiences in teaching or using logarithms. On the initial pretest, administered before the research group had engaged in any development work with her, Paula was able to decode a logarithmic expression such as $\log_2 5 = x$ into its exponential equivalent, $2^x = 5$. She was also able to locate the change of base formula in her course text book and translate $\log_2 5$ into $\log_{10} 5/\log_{10} 2$, but did not know how to use her calculator to evaluate the resulting expression. Her abilities to use the **Fill** command on the table window in Function Probe enabled her to solve some of the problems,[4] although

[4]The pretest problem: Suppose $175 is deposited in a savings account earning 9.5% annual interest. When will the original deposit be doubled? Paula set up three columns: x=initial value, y=.095x, and sum = x+y. She entered the initial value in the first column in her table. After Function Probe computed the interest and the resulting sum,

she was unable to formulate any equations describing the functional relationships. A second pretest was administered about ten days after the exponential development work, but prior to Paula's teaching of the unit or additional development with logarithms. At this point Paula showed considerable progress, generating exponential functions from contextual problems and working with geometric sequences. By the end of the exponential unit her knowledge growth was substantial. On the posttest she demonstrated, for instance, multiple methods for computing $\log_b a$, using all of the windows in Function Probe, as well as the change of base formula[5].

b) Generative power of subject matter expertise.

Aspects of the teacher's newly developed knowledge of multiplicative growth took on a certain robustness. The first few lessons of the exponential unit introduced the concept of geometric sequence and geometric mean, then used these concepts to develop the notion of exponential function as isomorphism between an additive world and a multiplicative world. Paula felt very comfortable and enthusiastic about the lessons and more comfortable with her own understandings of the mathematics. This led her to be free to generate her own ways of explaining concepts, of posing questions, and of creating examples for class discussion. She explained:

> I just feel like I understand this material so much better so far, so that what I am doing in class is coming out of my own words, and my own thoughts; and before, I had to struggle to think, what should I do next, what questions should I ask next? What point am I trying to get across to the kids? But since we've started with this material, that's another thing that made me so incredible [sic], is when I received from you the teacher copies, with the extra remarks on them, those would be the exact things that I would

---

Paula typed the result in her first column, repeating the process until she had reached a sum that was approximately double the original amount.

[5] The posttest problem called for demonstrating multiple methods for determining a solution for $\log_2 60 = x$. Paula estimated the value of $\log_2 60$ to be between 5 and 6, then used progressively finer fills in the table window to approximate x. She also graphed $y = \log_2 60$ in the graph window and rescaled appropriately to get an approximation for x. In addition, she was able to apply the change of base formula easily, as well as build a base 2 button in the calculator window and input 60 to determine x to the desired precision.

have done! So that, it's like, I'm clicking, it's like, I'm coming up to you guys, saying, I'm figuring these guys out . . .!

(1/24/90)

This *generative power* of her newly-constructed knowledge is evidenced in other ways. For example, Paula started to see connections between some of the concepts she was working with in the precalculus class and her geometry lessons. She described how the current lesson in geometry dealt with the notion of geometric mean as related to the common leg between two similar triangles:

And the way the book explains it, they just say it's a common leg of a triangle, and it's a proportion. So find the geometric mean of 1 and 7. So [the students] have to be able to set up a proportion and then they give them a picture of a triangle, a right triangle, that has two right triangles inside of it, and they give a leg that's common to two of the three triangles, and that's the geometric mean . . . so what I did today was, instead of just telling them, this is what you do, today I said, well let's draw a sequence on the board, because the kids were saying, what's a geometric mean, is it in the middle? And I said, that's an arithmetic mean, that's what you're used to hearing, what's the arithmetic mean of 1 and 3, and they all said 2, so they understood that, but I said geometric mean is not exactly in the middle . . . .

(1/29/90)

She then developed with her class a geometric sequence with multiplier of 4 [1,4,16,64,256, . . .] and developed the idea that a geometric mean between any two consecutive terms would be the initial term times 2. She related the geometric means developed through the sequence representation to both the algebraic representation of the mean proportional ($a/b = b/c$) and the geometric mean for the right triangles:

. . . so I went through and we filled in, I had my sequence, I filled in all the geometric means, and then I went back and we did a proportion, like the book had done, and um, using these numbers from my sequence over here, now if this is the geometric mean, let's put this into our proportions, between this number and that, and then we went to the triangle, and now let's look at this geometric mean in terms of this triangle, so there were three visual pictures, I would have never used before . . . never! Because I never

> understood the transition from one to the other . . . .
>
> (1/29/90)

c) Critical moments and recovery

Even with a clearly sequenced and carefully detailed lesson plan, Paula's occasional difficulties with subject matter led to critical moments in instruction. For example, during the lesson which was designed to introduce the concept of fractional exponents, Paula seemed to be unable to retain the agenda for the lesson. She reviewed the computer virus problem wherein the amount of memory taken up by a computer virus is being multiplied by a factor of nine every hour, then launched a discussion about what kind of growth could be envisioned after half an hour. But instead of extending the discussion and helping the students to see the link between half-hour increments being related to memory loss in multiples of 3 (or $\div 9$), she dropped the discussion, and went into a demonstration of algebraic procedures to justify $9^{3/2} = 9^{1+1/2} = 9 \cdot 3 = 27$. She knew her explanation was puzzling to the students, and she asked them if they felt "totally lost." Unsure of what direction to take, she assigned them the task of determining the equation which would represent memory loss every half hour, and then ended the class a few minutes early, telling them, "I want to pick up more on this because I wasn't happy with the flow here" (2/7/90). In a later interview she revealed that she felt the end of class was disastrous. She explained that, although she felt she understood the material, she was unsure of how she was

> . . . going to get the material to the student. How I was going to illustrate what I was doing, right here. I know exactly what I was doing, because when you multiply exponents it means you've got $9^{1/2}$ cubed, or $9^{3/2}$ . . . that was what I was trying to get at, but that's really, when I made this [her notes] out, I didn't really even think about that, I just did it, because I just knew it, and all of a sudden the class was going [in the wrong direction] .
>
> . . . I lost sight of what my goal was, and therefore I ended up stumbling, and I thought it was so clean cut and so clear, and then all of a sudden, smack right in the middle of class it was like, where was I going to go with that, and it was very frustrating . . ..
>
> (2/7/90)

Having tc ˋ˴ ˀelop her understandings of the mathematics and simultaneously draw from her newly constructed knowledge base in order to teach her students was a continual challenge. There were times, such as this one, when Paula was forced to draw on previously developed curriculum scripts and explanations. Her subsequent reflection on this and other critical moments provided opportunities for understanding how to extend and modify her existing instructional schemes and coordinate them with her own deeper subject-matter understandings.

We believe that the issue of recovery is a key issue in creating substantial curricular change. No teacher can revise her/his understanding of mathematical material and immediately translate that into effective instruction. Learning to hear the diversity of student ideas will necessarily result in moments when a teacher feels lost or uncertain of how to relate a student's ideas to her/his own. Allowing these instances to occur and learning to recover from the embarrassment or awkwardness one feels when one is supposed to be "expert" is an important part of teacher development. We encourage teachers to learn to tell the students clearly that they need to consider the idea, and to give themselves out-of-class time to do this, bringing the ideas back in subsequent classes. However, it is really exciting to witness the first times a teacher is able to work with the class to effect a recovery during the class. It demonstrates the flexibility and openness with which the teacher is beginning to approach teaching mathematics.

d) Multiple Representations

In September, Paula claimed that the table representation on Function Probe was probably the least helpful representation for her, even though she saw the value in having several representations side by side on the screen. Her preference at the beginning of the linear functions unit was to work with the equation representation, $y = mx + b$. As the year wore on, Paula found herself using the table window more and more as an entry into problems she would not otherwise be able to solve. By the end of the exponential unit Paula was able to move comfortably between windows and coordinate actions among them, developing function buttons in the calculator window and sending them to the table window, and using the table window and the **Fill** command to generate patterns, test conjectures, and approximate solutions. The graphical representation seemed to be her weakest and least-utilized representation throughout the innovation. Her increase in competence with Function Probe was evident in her increased use of the PC viewer to demonstrate actions in and between windows during whole class discussion. There seemed to be a smoother flow between whole class and group work, between board work and computer work, than in the beginning of the year.

Conceptions of Mathematics Teaching

Paula showed growth in her conceptualization of constructivist teaching, moving from a perspective of mastering teaching techniques to one of developing metacognitive awareness of her own mathematical processes.

a) Mathematics for teaching: a metacognitive awareness

Paula felt more confident of her role in January than she did in September. Convinced of the value of small-group learning, she nevertheless saw a need to combine group work with some whole group instruction, in order to find that important balance between "direct instruction" and "no instruction." In September, many interviews with Paula seemed to focus on issues of grouping, class management, and grading. In January, interviews were more subject-matter focused. Paula realized that she needed to discuss the subject matter that she was grappling with on multiple levels. As an individual learner, she had to construct her own meanings for the mathematics in light of her current mathematical understandings. As a teacher, she felt the need to understand her ways of analyzing and approaching the problems at a metacognitive level of understanding, rather than being concerned with merely arriving at a satisfactory solution. And in addition, she felt it was important to understand the ways that the research group made sense of the mathematics themselves, through the questions they asked and the ways that they probed her understanding of the mathematics. She noted:

> In early January, those three days we're going to do problem solving, I want to really think about the way I'M thinking how to solve the problem, but I also want to hear what my mentor, or whoever is sitting there, what kinds of questions they're asking ME, to get me to think even more clearly about the problem . . . because . . . that's going to be me, over there, pretty soon, and I just want to, I keyed so much this summer into getting the answer to the problem, that I want to kind of step back a little bit, and yes I need to work through the problem and to be familiar with it, but I need to look at this whole problem and take in the whole, all these different viewpoints, and really understand it, and I really think I understand the problem now, that we've already

> done, because of going over them in the class presentation, I think I really understand them much better than I ever did; before we ever did that, I THOUGHT I did, but then you REALLY have to understand it when the students start changing those equations around a little bit . . . .
>
> (11/8/89)

Paula approached her own learning of the subject matter on different levels. At the same time that she struggled with the substantive content of the exponential unit, she was also developing a sensitivity and awareness of her own problem-solving processes, as well as addressing the issue of how to transform the subject matter in a way that would be meaningful to her students. When asked to compare the development process for linear functions which was held during the summer with the development work on exponential functions which had been held in early January, she stated:

> I also didn't know exactly what I was going to do with this material . . . is this it, is this what I'm going to give to the kids, what am I going to do with this? I didn't know how the group work was going to work, I didn't know any of that, and now, now we have a format, we know what to expect from the kids, I think there's something . . .. Last summer I would just sit there, and I didn't want to make a fool of myself, because I didn't know you guys that well, and I wanted to make sure that I knew how to solve the problems, so I was really concentrating on the solutions to the problems, and this time I was more relaxed, and yes, I wanted to know the solutions to the problems, but I kept constantly stepping back as a teacher, seeing the role that [you were] playing . . . asking me questions, how you made my thoughts move, so I knew to key in on other parts of the conversation besides the actual problem involved.
>
> (1/24/90)

b) Teaching techniques versus deep subject matter understanding

In an interview with the teacher conducted in early November by Linda Knapp of Apple's ACOT division, Paula described the innovation as "teaching technique," and relegated a lesser significance to the treatment of the subject matter. Later, in January, she commented on the interview, maintaining that her views on the importance of the subject matter had changed, and that her own understanding of "the whole math world in itself," was very crucial. She said:

> I would answer differently now . . . at the time it was [teaching technique], but now, I also think it's the material . . . teaching technique is very, very important, but I could have never created this material, or this understanding of the material, the development, how one mathematical concept can lead into another, and how a mathematical concept can be multirepresented, . . . in picture form, in equation form, in using just 'a bunch of numbers together' form . . . ."
>
> (2/1/90,)

Conceptions of Mathematics

When asked about her conceptions of mathematics, Paula said that she saw mathematics as two (mostly) separate subgroups -- school mathematics and the mathematics actually used in real life. While she saw the mathematics she taught as a "foundation for higher learning" and necessary for certain occupations, she maintained that "I still put that in a separate category from the math that I encounter in my checkbook, or figuring out how much paint I have to buy" (9/6/89).

Paula did not rate herself highly as a mathematician, but was confident of her abilities to teach mathematics; in fact, she joked that her slowness to grasp concepts was actually an asset to her teaching, making her a much better teacher than a mathematician would be (9/6/89). She confessed that she was really "hung up on story problems," and that caused her a lot of concern about the project. Her lack of self-confidence in her ability of solve word problems had led her to avoid them in her own instruction as much as possible, but at the same time she felt that problem-solving ability could be acquired, and she did not want her students to share her feelings toward word problems (9/6/89). She also acknowledged that she had felt very insecure and threatened during the summer development week, because she saw herself working at a very different level mathematically than the members of the research group, and even the ACOT science teacher.

Later on in the year, Paula seemed to change some of her previously held views about mathematics and her own mathematical abilities. Using the analogy of sets, she described her world of mathematics in the fall as a finite set, anything found in a math book. This conception had changed for her, and she now saw mathematics as an infinite set (1/28/90), and humankind as mathematical by nature. In the past she believed that there could be several different ways to solve a problem but only one way to view the problem, and "now I realize how open ended math problems can be . . . a teacher should let their [sic] students develop these natural interpretations and build on them, not shut them out and say that that particular way of thinking is incorrect" (1/28/90). After a development session on site with members of the research team, during which Paula took an active role in the discussion of iteration versus recursion, she was asked to comment on her own feelings about the experience. For the first time, she expressed a confidence about her own mathematical abilities:

> "I keep categorizing you people as mathematicians . . . I couldn't sit down and do and tear and build and create, like they have, and I know you keep reassuring me like you have, but . . . that was my first experience to live what you've been telling me, to realize that, gosh, they really aren't getting this from a book! They really aren't! . . . and I was a part of it! It was so exciting, I thought, maybe I could go on and do some more things like this on my own, maybe there is a possibility, I mean, it's given me more and more confidence . . .."
>
> (2/1/90)

Part of the outcome of this development work was Paula's creative use of MacDraw software to represent changes in areas represented by quadratic functions. She was able to combine her technical knowledge of the software with her newly constituted knowledge of quadratic functions to create a challenging group project for part of her quadratics unit.

Student Methods

In this section the dialectic between Paula's ability to recognize the legitimacy of student methods and her own growth in subject matter understandings is described, as well as her recognition of the importance of student autonomy in the problem-solving process.

a) The development of pedagogical content knowledge

One of Paula's greatest strengths as a teacher is her ability to listen to students. She acknowledges there is an even greater need for deep and flexible understanding of the subject matter to teach within a constructivist framework for instruction, and that asking the right question can be harder than supplying the right answer. She has learned to stay longer with groups, rather than moving too quickly about the room. She encourages students to coordinate actions within and between windows to test their conjectures and verify their solutions. For example, to verify an equation she looks for a data column using the **Fill** command and a second data column with identical entries, but generated using an equation.

Another development has been the change in her ability and desire to reflect on her teaching through the perspective of subject matter. Like the other teachers who participate in the ACOT project, Paula has kept an audiotaped journal of her thoughts and feelings about being a part of the ACOT experience. She has not used this medium, however, to reflect upon her own mathematics or the ways in which her students make sense of what she is teaching. She has spoken of her past practice of "shutting down" her thought processes after the end of a class. She describes the practice as a survival technique, one that she developed as a response to years of teaching six classes a day, and not having the inner resources to look ahead as well as to reflect back.

That practice, she feels, has changed for her: "I have a strong need now that I'm really into the way they're thinking, to go back and try to view now the lesson through their point of view" (11/10/89). By the end of the linear functions curriculum she had begun to take notes on student solutions and comments when they made group presentations in class. By taking notes of differences and similarities across groups, she was able to organize her own thinking about the different interpretations and solution strategies, which in turn affected her planning for subsequent lecture and discussion. Student insights and comments, she felt, were opportunities; she noted that "I need to weigh that, and value that, and think about that, I need to let it remain in my mind a little bit longer than I am so I can process it . . . is it going to lead into any new thoughts or questions . . ." (11/8/89). By listening to students construct their own ways of viewing and solving problems, she has come to recognize her own growth in mathematical understandings. For example, after student presentations on the parking garage problem which was the last group problem in the linear functions unit, she realized that she " . . . understood much better. Much better. I mean there were some things in there I didn't pick up until they [presented their ways of thinking about the problem]" (11/8/89). She recognizes that by listening to the students she is building up a repertoire of student methods, conceptions and misconceptions, and thus is more able to address issues in her instruction that previously she might not have thought to bring out.

b) The need for subject-matter expertise

Paula began to realize the importance of her own subject-matter understandings, and the growth of those understandings which has been made possible through her reflection on student ways of knowing and understanding her lessons and problems. At the same time, her ability to recognize the legitimacy of alternate approaches and ways of making sense of the mathematics by her students has been enhanced by her own development in mathematical understanding. When she was having her own struggles with the mathematics she was less satisfied with her classroom instruction. There were times when she passed over what appeared to be legitimate concerns of the students or positive insights, often judging their comments as only tangential to the lesson at hand. For example, at a critical part of her lesson on extending the domain of exponential functions to include all real numbers, a student expressed some concern about how to go about understanding what could be meant by raising 2 to an irrational power, such as π. Paula, conscious of the time press and her own agenda, responded that one would have to simply raise the base to an approximation of π, rather than using this concern as a springboard into the core of the lesson.

c) Student autonomy

It has been important for Paula to allow students the opportunity to share their insights and solutions with the rest of the class. She has allotted class time to group presentations, both during the linear and exponential units and at other times during the school year. She not only asks the students to share their solutions, but to be able to field questions about their thought processes, choice of representations, and conclusions, from both her and their fellow students. By acknowledging alternate ways of approaching and solving problems, and allowing students the opportunity to share these approaches, Paula feels that the students have developed a healthy autonomy and taken on a greater responsibility for their own learning.

For the lily pad problem[6] in the exponential curriculum, Paula decided that it would be more efficient for all groups to designate the independent variable as "time past," with initial value for t being 0, not 1. One student protested, saying that they should be allowed to make their own choices for how to designate variables and set up their tables, as long as they could justify their choices and explain the meaning of their answers. Paula allowed this freedom of choice, and expressed her delight over this incident in a subsequent interview. She saw this as a clear indication of the increased confidence the students had in their ability to solve problems and become independent learners.

Routines

Paula adapted new classroom routines and subsequently modified them as the year progressed and as her familiarity with the software tool and the curriculum increased. In this section we describe her use of groups in classroom instruction, changes in her assessment practices, and issues of pacing.

a) Use of groups

While Paula continued to make use of group work during the school year, she became less focused on group structure and allocated less class time to group activity. In September, almost all class time was spent working in groups of three or four, where students were assigned specific roles such as keeping the group on task, keeping a written record of the group's activity. Paula tried to keep whole class instruction to a minimum, even at the expense of efficiency, such as giving directions or making announcements which would be relevant to the whole class. By January, group work took a smaller percentage of class time, and was intermixed with individual student work and whole class instruction. Although students were initially assigned to groups, the assignments were not enforced, partly because a high rate of student absenteeism necessitated frequent regrouping. At the same time, Paula recognized the need for the subject matter to drive her class management, rather than conversely. Reflecting on the linear unit in September,

[6]The problem: On May 1, you discover a 5-acre pond in the country with a single lily pad of area one square foot. The next day, there are 2 identical pads. If the number of lily pads double each day, and none die, will you be able to go swimming on June 1 without entangling yourself in lily pads?

she felt that she had made group work her chief priority over the subject matter, and that now the priorities had been reversed, with the mathematics driving the grouping and general class management decisions.

b) Assessment

Paula made changes in her assessment practices as the school year progressed. In September, much of our discussions on assessment revolved around how she was going to assess the group work. She made changes in her grading policy for each one of the contextual problems, which made her as uncomfortable as would have staying with an unsatisfactory grading policy. As an example, one model she used for grading groups broke the group grade into three parts: her assessment; the group's written assessment of itself; and the group's presentation of results. In January her grading focused much less on group work and more on individual work, with grades being given for a mid-unit test and posttest, quizzes, homework, challenge problems and worksheets, and with group grading making up a much smaller percentage of the group's total grade.

Students were rewarded for their class participation through a point system. Although she believed that students were motivated by the contextual problems, Paula awarded points for good questions, comments and answers, because she felt that the students would not make the effort otherwise. Sometimes she handed out point sheets, a coupon of sorts which could be signed by the student and redeemed for points. Homework became a critical issue for the exponential unit, because the students often failed to complete assignments and were not prepared for the next lesson, leaving Paula exasperated, especially since her own out-of-class involvement was so great.

Paula's tests, quizzes and handouts reflected her own awareness of the the major goals of the unit. The issues that became important to her in her own development became issues important to include on written assessment measures. She showed a remarkable ability to design novel nonroutine questions to ask of the students, which required them to coordinate multiple methods and/or representations. In addition, problems no longer needed to conform to the exact format of previously assigned homework problems or examples worked out in class. She explained that in the past she would never have asked something on a test that she had not gone over first in class. This was true even for the wording of the problem. She had expressed concerns about the pretest the students took at the beginning of the school year, and suggested that perhaps some of the difficulties students had in solving the problems on the test could be attributed to the wording of the problems, which was different than that of the text book. She explained:

> In the past, . . . I have always phrased it exactly the same way the book's phrased it, and I've always been very leery about deviating from that way of asking the questions, because I've felt that's the way they're comfortable with hearing it asked, and now I'm really, as you can see, I'm really leaning away from that, I'm not taking that totally off the test, but there's no reason why they can't do some applications of what they've learned, why I can't expect that of them and test them, and grade them on it . . . in the past, if I would have done any of that it would have been an extra credit problem.
>
> (11/10/89)

c) Pacing

Pacing is an issue which Paula feels she has not resolved to her satisfaction. Because students often failed to complete homework assignments and prepare for class, Paula regularly found herself repeating material from previous lessons. Her own struggle to work out the mathematics for herself and, at the same time, transform her newly constructed knowledge for teaching was difficult and time consuming, both in and out of class. She acknowledges that her own development process in the understanding of the mathematics itself, as well as in her understandings of the mathematics from a pedagogical perspective, had the effect of slowing the classroom pace. At the same time, the commitment of the students to their studies waned considerably as they approached graduation, which meant little time spent outside of class on studying and doing homework, and she often found herself reviewing concepts that she felt they should have already learned, but did not. She continues to struggle with this issue, as she reflects on her teaching, and finds no easy solution.

Students

As the students used Function Probe to solve contextual problems, they developed their problem-solving abilities and expanded their conceptions of functions. After presenting a summary of students' uses of multiple representations of functions, specific test results and interview results will be discussed.

Using Multirepresentational Software

Table Window. The students were able to use the table to organize the information of a problem and to generate information about the table data such as differences or ratios between entries, although they often had difficulty identifying the variables clearly. They could coordinate the actions of filling two individual columns with the sense of covarying rates of change for the two columns, but they often had difficulty coding the relationship between the two columns algebraically.

Many students felt comfortable using Function Probe to test guesses and hypotheses. For example, one student used the **Ratio** command to verify his guess that a sequence was geometric; another student built an algebraic equation, checking the computer-generated equation values against her expectations. Many students preferred working in the table window and were able to solve entire problems within this representation.

Graph Window. Results of classroom observations show that many students used graphs as a secondary representation after first using the table or calculator to gain entry to relationships within problem information. However, they were able to link the numerical data from tables and calculators with the graphical representations in understanding such concepts as slope and rate of change of functions.

Many of the students developed a strong qualitative sense of functional relationships that is not as easily developed when students examine hand-drawn graphs. By quickly and efficiently graphing functions and presenting them to students as objects for examination, graphing software allows students to concentrate on global features of graphs, such as shape, direction, and location, and thus leads to understandings based on features which distinguish between classes of functions.

Other features of the graph window proved to be important learning tools for the students. They changed the scale of the graph window to situate their graphs on appropriate axes and to examine the extreme points of graphs. They used the translating, stretching, and reflecting features to coordinate the algebraic and graphical forms of functions. They used the point-locator to find points of intersection between graphs. Students soon became facile with these features of the tool and were able to spend more time in solving problems and discussing their solutions with others.

Calculator Window. The students used the calculator window extensively to build operational and numerical forms of functional relationships, as well as to carry out basic calculations. For example, students used repeated multiplication to model the actions of a computer virus creating an exponential function, and they then built algebraic equations to model the functional relationship of this situation. Textbooks at the high school and college levels, by presenting students with full-blown algebraic representations of functions, rarely allow students to develop this important ability.

The Role of a Problem-based Curriculum

In addition to the role of multiple representations on students' conceptualizations of functions, a significant role was played by the contextual problems of the curricular units designed for this study. The students remembered key aspects of many of the problems over several months, and they developed a sense of a "prototypical" linear function or exponential function from certain problems. They often referred to these prototypes by name when encountering similar situations in later problems.

Pretest and Posttest Results

The linear functions pretest, containing 20 multiple-choice items, four open-response, nonroutine items, a sentence-completion task and one problem-solving problem, was administered on September 1, 1989. A modification of this exam was administered as a posttest on October 13 and 16, 1989. A comparison of the results of the exams shows significant improvement for the entire class and for many individual students. Table I presents the results.

Linear Multiple-choice Items. For the twenty multiple-choice items, the mean on the pretest was 6.7 items correct (s.d.=3.4), while the mean on the posttest was 10.6 items correct (s.d.=3.9). This represents an average gain 62% and indicates a significant change (match pair t=5.9, $d.f.$=22, $p < .0005$). Twenty out of twenty-three students showed improvement in score from the pretest to the posttest; two students showed no change; and one student showed a loss.

Linear Problem-solving Items. The mean on the linear functions problem-solving pretest was 36.0% correct (s.d.=20.9), while the mean on the posttest was 48.3% correct (s.d.=24.1), a 34% gain. The problem-solving items were scored by two independent scorers, graduate students at Cornell with familiarity in grading such problems. Interrater reliability on the linear functions test was 98%.

Pretests for the exponential functions unit were administered on January 5 and 10, 1990. The first part contained eleven multiple-choice items, six open-response, non-routine items, and six matching items. The second part contained a problem-solving problem on which students were allowed to use Function Probe. Similar posttests were administered on March 15, 16, and 19, 1990. Results of the exponential functions pretest and posttest for twenty-two students show significant improvement for the entire class and for most of the individual students. (See Table I)

Exponential Multiple-Choice Items. For the eleven multiple-choice items, the mean on the pretest was 4.4 items correct (s.d.=1.8), while the mean on the posttest was 6.2 items correct (s.d.=2.2). This average gain of 41% is significant (match pair t=3.8, $d.f.$=21, $p < .001$). Results for individual students show that sixteen out of twenty-two students showed an improvement in score from the pretest to the posttest; four students showed no change; and two students showed a loss.

Exponential Problem-Solving Items. On the problem-solving parts of the exponential tests students' scores improved from 60.8% correct (s.d.=18.6) on the pretest to 73.1% correct (s.d.=12.3) on the posttest, a 20% gain. Interrater reliability was 100% on the grading of the exponential functions test results. Table I presents the results.

TABLE I

Pretest and Posttest Results : Average Percentage Correct

	Pretest	Posttest
Linear Functions		
Multiple-choice	33.7%	53.3%
Problem-solving	36.1%	48.3%
Exponential Functions		

Multiple-choice	39.7%	56.6%
Problem-solving	60.8%	73.1%

While these results are positive, they should be examined in light of several factors which may have affected the outcomes. First, the linear pretest, linear posttest, and exponential posttest were administered by the classroom teacher. Even though she had been given specific instructions for administering the tests, we cannot attest to how the actual administrations took place. We do not know the exact conditions under which the students were placed nor the instructions given. Posttest scores may have been higher due to the fact that the teacher modeled daily challenge problems (part of the ongoing classroom routine) on the problems from the pretests (even after she was asked not to do so). Also, the students were told that the posttests would affect their grades in the course. And, according to the teacher, the exponential posttest was administered over several days due to scheduling problems. Also, as in any such situation, one can never know whether students have done "their best" on either the pretests or the posttests.

From the outset of this study we did not intend to use the testing data to draw conclusions about the effectiveness of the innovations or to make generalizations from this group of students to others. The quantitative results of the tests serve mainly to document that the skills of these students were not adversely affected by the implementation of this study. We believe that clinical interviews are more effective in evaluating and analyzing students' abilities and conceptions than are tests, especially multiple-choice tests. Students' uses of multiple representations to understand functions are well documented in the other data sources and are discussed below.

Student Interviews

The Concept of Function: Student Conceptions

Students' formal definitions of the concept of a mathematical function, as well as the concept images they held, varied widely for many of the students depending on the task on which they were working (defining, classifying, or constructing). The following categorization scheme is based on analyses of all of the instances of students' uses of a description or a definition of the concept. The categories reveal changes in definitions across time and allow one to see interesting patterns in terms of which definitions were elicited by which tasks. The task on the pretest and posttests was to define the word 'function'; the task in the interviews was to classify tables and graphs as functions or nonfunctions.

On the linear pretest and posttest, and again on the exponential posttest students were asked to write their definitions of function as a sentence-completion task.

Test Item: *How would you define the concept of a function as we use it in mathematics? Complete the following:*

In mathematics, a function is . . .

Students' written responses to this test item differ over time due to classroom instruction, their experiences with using Function Probe to solve problems, and, to some extent, due to practice effects.

TABLE II

Categorization of Student Responses:
"A FUNCTION IS . . ."

Category		linear pre 9/1/89	linear post 10/13/89	expo post 3/16/90	ints 9/89	ints 2/90
A function is:		n=23	n=23	n=22	n=8	n=15
I.	a procedure used in solving a problem	4	3	1	0	0
II.	an equation	7	5	4	2	1
III.	a graph	3	11	9	0	0
IV.	symbols that represent something	1	1	0	0	0
V.	a solution to a problem	1	0	1	0	0

VI. verified by the vertical line test	2	7	3	5	9
VII. ordered pairs	0	0	5	0	0
VIII. pairs w/ unique first element	0	5	2	1	8
IX. a covariation relation	0	0	0	0	7
X. any correspondence	0	0	0	0	3
XI. corresp. w/ unique 2nd term	1	2	0	2	5
XII. 1-to-1 corresp.	1	1	1	1	1
XIII. corresp. w/ unique 1st term	0	3	7	2	11
XIV. has a consistent pattern	0	0	2	2	7
No Response	8	3	2	0	0
Total Responses	28*	41*	37*	16*	52*

[* The number of responses is greater than the number of students due to multiple responses.]

Observations based on these results include the following: First, the number of students responding to this test item increased with each administration. Fifteen out of 23 students responded in September and twenty out of 22 responded in March. Second, the increase in the number of different responses offered reveals that students were incorporating multiple ideas into their concept images of function.

The algebra-based definition of a function as an equation was the predominant definition in September, but it was surpassed by graphically- and numerically-based definitions in later administrations. As students had more experiences with functions in various representations they moved from a model of a function as a procedure to one of a function as an object or entity.

The covariation definition of function did not appear in any responses on this task, and the idea of a function as a correspondence (the modern definition) appeared rarely. These responses were some of the predominant responses in the interviews, however. It appears that students' concept images of functions as pairs of covarying or corresponding numbers is rooted in the tabular and graphical representations of functions rather than the algebraic.

Asked to classify tables and graphs as functions or nonfunctions in interviews in the winter, students' responses were now based on (1) patterns they found in the tables and graphs (including a covariation relationship between elements), (2) the use of the Vertical Line Test, and (3) the notion of a correspondence with a unique first element.

Interview Protocols

The following section presents students' actual actions and words from class, and interview problem-solving efforts, and it concentrates on two specific aspects of the course: mathematical modelling and the concept of rate of change.

1) Mathematical Modelling

Constructing a mathematical equation from a contextual situation, also known as mathematical modelling, is an important use of applied mathematics that is minimally included in the secondary mathematics curriculum and is often completely ignored in the secondary classroom. For this project most of the problems written for the units on linear functions and exponential functions involved contextual problems in which students were to construct mathematical functions from contextual situations. An example of a problem from this curriculum is The Parking Garage Problem. This problem, the final problem written for the unit on linear functions, is intended to lead students to examine transformations on graphs (translations and stretches) and the coordination of the algebraic equations with such transformations. The Parking Garage Problem is purposely open ended to allow students to choose their own methods, to decide which representations to use, and to invite discussion about which option is 'best.'

The Parking Garage Problem. The mayor of our city, in an effort to encourage shopping at the new downtown City Center, has called together a task force to come up with a revised schedule of parking rates. Here are the four options of parking rates at the City Center parking garage. Option I: Pay 35¢ for up to, but not including, the first hour; pay 50¢ for up to, but not including, each additional hour. Option II: Pay 10¢ for up to, but not including, the first hour; pay 50¢ for up to, but not including, each additional hour. Option III: Pay 35¢ for up to, but not including, the first <u>half</u> hour; pay 25¢ for up to, but not including, each additional <u>half</u> hour. Option IV: Up to the first hour is free. Pay 75¢ for up to but not including, each additional hour. Which option should the mayor choose? Explain your answer.

This problem followed the introduction of the greatest integer function by the teacher through a teacher-developed example using birth age and chronological age. The students began to work on The Parking Garage Problem in small groups for ten minutes at the end of class one day and got back into their groups the next day to continue their work on this problem. The protocols presented are from taped classroom observations conducted in October, 1989.

Working together, Chip and Katrina have their own version of "cooperative effort:" Katrina controls the mouse and Chip uses the keyboard as they work on Function Probe. Based on his work on previous problems in the unit, Chip quickly figures out the linear form of the equation for the second option directly from the situation although he does not include the greatest integer function. He writes y=.50x+.10 saying, "The equation for this is simple." He does not explain how he arrived at the equation, and, when Katrina immediately tries to write the equation for the first option, she reverses the parameters.

Katrina: This one's gonna be the same thing. 35x plus 50. Right?

Chip: No.

Katrina: Why? It's the same thing. You pay 35 cents. 50x+35, isn't it? So I had them backward.

Chip: The equations are -- I think the first equations are linear. Like mx+b type things.

As class time runs out they write equations for the third and fourth options, and as they continue on the problem the next day Chip expresses satisfaction with their equations.

> Chip: This will be easy. We'll get this finished real quick.
>
> Katrina: Are we sure that these are gonna work?
>
> Chip: No, but we'll find out soon enough. Yeah, I'm sure they'll work. I have no doubt in my mind. At least the first three 'cause if you substitute in, it works, so --

At this point they have only one of the four equations built correctly. The other three equations lack the greatest integer function notation, and one of those also has an incorrect argument of the function. (For Option I they have y=.50x+.35 rather than y=.50[x]+.35. For Option II they have y=.50x+.10 rather than y=.50[x]+.10. For Option III they have y=.25x+.35 rather than y=.25[2x]+.35. And for Option IV they have the correct equation, y=.75[x].)

Hoping to create an awareness that their equations are not exactly correct for the situation, the observer asks what Chip and Katrina plan to do with these equations, and they discuss their expectations for the graphs of their equations:

> Katrina: We're supposed to get step graphs, but we don't know if this is gonna work.
>
> Chip: Not for all of them.
>
> Katrina: Or the greatest integer function. That's what she [the teacher] called it.
>
> Chip: I think it's just for the last one that you'll get that because these I don't think you will. I know you won't because that's a linear equation which --
>
> Katrina: It works.
>
> Observer: But does it satisfy the situation?
>
> Chip: Yeah.
>
> Katrina: Yeah.

Observer: What if you park for an hour and a half?

Chip: You'll pay the same that you pay at one hour.

Observer: So that's not linear then.

Katrina: So then all our equations are wrong.

Chip: So wait. Could you just go like this then?

He writes y=[.50]x+.35 on paper, rescales the graph window, and then enters a different equation, y=.50[x]+.35, into the graph window as a new equation to be graphed. On Function Probe the brackets are read as parentheses and the graph of y=.50[x]+.35 appears as a straight line without steps. Katrina remembers from a previous class discussion that to enter the greatest integer function on Function Probe they are to enter y=floor(x). They enter y=.50floor(x)+.35 into the graph window and see the appropriate step-like function. They then enter the other equations using the "floor" function and print out the graph window with four graphs superimposed on the same axes.

After this work Katrina is able to summarize to the observer how the equations were built:

> See it's 35 cents for up to but not including the first hour. No matter what, you're gonna pay 35 cents. So that was our adder. The point thirty-five. And then it was just pay an additional 50 cents per hour and whatever you park it's gonna be for each hour. If we made a table, x would be the hour. Just like all the other problems we had. We had a set -- Like The Club Trip Problem. We had something set which you had to pay no matter what. And then you had your additional [charges per day].

For Katrina the slope-intercept form of a linear function, y=mx+b, now has personal meaning. The 'adder', b, is the initial amount in a situation; the 'multiplier', m, is the amount that accumulates per unit of time, x. And while the language of 'adder' is not her own invention (it was developed in a previous class discussion after working with the Function Finder© software), Katrina finds it useful in talking about how she is coordinating this situation to the mathematics that can represent it. This is a strong coordination of the actions involved in the situation, the operations used to model the situation, and the algebraic representation. Her reference to The Club Trip Problem, a problem from the preceding week, shows that she has coordinated the actions and operations in a general way rather than for a particular situation.

Katrina and Chip are pleased to be finished quickly and they are not aware that their third equation is not correct for the variable x defined in hours. It is not until the class discussion a few days later, in which groups of students present their individual equations and graphs and their choice of the "best" option, that this becomes an issue. This issue is discussed as a case study in the results section on small-group problem solving.

In the class discussion of The Parking Garage Problem which followed at the beginning of the next week, Chip also had the opportunity to articulate his understanding of this function. The class was looking at the equation and graph of y=.25[x]+.35 (Katrina and Chip's equation for Option III) on an overhead screen, and one of his classmates asked why .35 is added each time. Chip immediately responds:

Chip: It isn't added each time. It adds it once.

Student: Why once?

Chip: It's like The Stone Path Problem.

Student: Why is it doing that?

Chip: Because when you start off you have to pay 35 cents.

Student: Right. But every additional hour -- Why does it --

Chip: The 35 cents is there from the start. Right here it's at 35 cents. So when you get over here it adds 25 and you still have the original 35.

Chip's sense of the function is that of an initial cost which is incremented at a constant rate per hour. He sees it in its y=mx+b form, but he has made sense of why that is the correct form in which to represent this situation. His reference to The Stone Path Problem, from four weeks prior, suggests that he, too, has brought away more from a problem than just the particular algebraic equation which represents the situation.

Chip's ability to quickly create the algebraic equations to represent The Parking Garage situation options may be based to some extent on the fact that this was the last problem in the linear functions unit, and students had been creating linear functions from situations for five weeks at this point. It may also be based on his ability to generalize from problem situations. In an interview at the beginning of the exponential functions unit, three and a half months later, Chip discussed what he remembered from the contextual problems of the previous semester:

> After we did the first one, and we finally got it set that this was the slope and this was the y-intercept, the rest were easy. 'Cause it was just the same problem with different numbers. You don't have to read the problem, just look for like -- see the two numbers and look at a little context and you know where they're going and you've got it. You're done.

Chip is not one of the students who had trigonometry as juniors, but he is one of the most active during class discussions and he is quick to come to conclusions in class.

The following excerpt, another example of mathematical modeling, is from an interview conducted in early February. The class had been studying geometric sequences after an introduction to exponential growth through a contextual problem. They had developed the formula for the nth term of a geometric sequence, had written equations for exponential functions, and had graphed such functions on Function Probe. The House Cooling Problem was used in interviews only; it was not written for classroom use in the exponential functions unit. (There were several other questions in addition to those stated here.)

> *The House Cooling Problem. With the heat turned off, and the outside temperature at a constant zero degrees Fahrenheit, the temperature of a house in the winter decreases such that each hour the temperature is 95% of what it was the previous hour. If the house is currently at 70^{0} F and it is exactly 6:00 p.m., what will the temperature be at 7:00 p.m.? What will the temperature be in 5 hours?*

Several students were unable to build an algebraic representation of a function for this problem during interviews, although they were able to answer the questions using a table or a calculator. Doug, on the other hand, after some initial exploration, is able to create the exponential function based on this situation. He attacks this problem by filling the first column of a table from 1 to 10 saying that these "x values" represent "the hours after 6 o'clock." Wanting to fill the next column using an equation, he enters several equations at the top of the next column, one at a time, looking for one that will yield the appropriate corresponding temperature values. He first enters y=.95*70 saying:

> I'm just trying to figure out how I should put the x values in. Since I don't -- Let's see." He then enters an x variable as a divisor to get decreasing values. Typing y=.95/x *70, he says, "If I multiplied times x, I thought -- For some reason I thought it would increase.

When he sees the table values from this equation he says, "Well, that's not right. . . . It can't drop down that much," and he changes the equation to y=.95x*70. He sees that this equation does yield increasing table values and is obviously not correct for the situation of a cooling house. Momentarily stumped, he pauses.

> Interviewer: Tell me what your goal is here.
>
> Doug: OK. Well, what I want to do is umm, I want to create a -- It almost seems like it would be what we've been working on lately. A sequential -- Maybe I just think that because we've been working on them so much lately. But it'd be sequential because the next term would be determined by the one before that. The previous one.

This important aspect of exponential functions has been developed by Doug through his work in class on a single contextual problem and on work with several geometric sequences. His intuition that this situation is comparable to the geometric sequences that they have been working on in class is correct, even though he is not able to articulate why at this point. Doug then realizes that he can build the column that he wants by using the **Fill** command on Function Probe. As he deletes the equation, the screen locks up, so he shuts down and reboots Function Probe. He again labels the first column 'x' and fills to 10, beginning at 0 this time instead of 1. He labels the second column 'y' and enters 70 in the first row. Zero hours after 6 o'clock now corresponds to 70 degrees in his table. He selects the **Fill** command to fill column 'y' to 55 having Function Probe multiply each term by .95 to create the next term. Asked for an equation for this column of numbers he then enters $z=70^{*}.95^{x}$ in the third column and sees that the values in columns two and three are exactly the same. Asked why x is the exponent rather than another multiplicand, Doug replies:

> Well, because -- Well, what this is doing -- Well, like the difference between -- The ratio between these -- It's multiplying each times .95. And you can see as -- All you're really doing is just -- You're multiplying the original term by .95 just again and again and again.

The action of the situation, the repeated multiplication by .95, is translated into the operation of exponentiation, and Doug is successfully able to build a correct equation for the situation, having coordinated the operations and the algebraic representation.

Since this interview took place early in the exponential functions unit it is especially informative as to how students use Function Probe to help them construct algebraic equations for functional situations. Unlike Chip's work previously discussed, this excerpt shows how a student who does not yet have a template for a situation can, nevertheless, reason out an equation intelligently and articulate such reasoning. Previously (Rizzuti & Confrey, 1988) we documented the struggles of a college student constructing exponential functions without the use of computer software. It was often a very difficult, and sometimes an impossible, task. As the excerpt above shows, a software tool can provide valuable assistance in this process.

2) Rate of Change

The next protocol data to be discussed are from interviews conducted during January and February, 1990. The problems involve exploring multiple representations of arithmetic sequences and geometric sequences as well as the linear functions and exponential functions of which they can be thought of as subsets. During whole-class instruction the class had discussed the functions created by mapping the natural numbers to sequence values and had graphed the resulting functions with the natural numbers on the x-axis and the sequence values on the y-axis.

In the data presented here, several students spontaneously generate the construct of the rate of change of a function in their explanations of why the graphs of functions behave as they do. The concept of rate of change was not brought up explicitly in the problems or discussions of the linear unit, nor in the problems or discussions of the exponential unit. Presented here are students' spontaneous, personal understandings of characteristics that are important to them in characterizing a function. In each case the rate of change concept is difficult for the students to put into words. There are many false starts and much fumbling of words, but the essence of the concept is part of their understandings, especially in relation to their interpretations of graphs. And because it is their own construction they are able to use it in different circumstances (exponential as well as linear situations) and over some length of time.

In an interview on January 30, 1990, Dan is given a piece of paper on which is written the first three terms of a sequence: 512, 3072, 18432, ... He is asked to determine which kind of sequence this is and to find the next three terms of the sequence. Just looking at the numbers Dan estimates that each value is the previous value multiplied by 6. Turning to the computer he types the numbers in a table column and uses the **Ratio** command on Function Probe to verify his guess. The **Ratio** command, which calculates the ratio between consecutive values in a column and lists the ratios in a new column, presents a constant ratio of 6. Seeing that his prediction is correct, he classifies the sequence as "geometric" and fills in three more values using the **Fill** command. He fills from 18,432 to ten million having Function Probe multiply each value by 6 to compute the next value. Asked for a prediction of the graph of the function created from this sequence, he says:

> All right. This would be linear because you're multiplying by the same -- No. It would not be. No. Take that back. Because 512 to 3,072 is smaller than from 3,072 to 18,000, so you're gonna have this slope again. It's gonna go to a real steep slope.

Dan's comparison of the differences between consecutive sequence values, the rate of change of the sequence, is his determining factor in predicting the nonlinear graph. At the beginning of this interview Dan could not correctly classify arithmetic sequences or geometric sequences, but he now has found a way to differentiate between arithmetic sequences and geometric sequences, and between linear functions and exponential functions, rather than relying on memorizing which type of sequence yields which type of graph.

In an interview on January 31, 1990, Rayna is given the first four terms of an arithmetic sequence, 317, 214, 111, 8, . . . , and she is asked to identify the type of sequence and find the next term. After subtracting pairs of values on paper she identifies this as an arithmetic sequence "decreasing by 103." Asked for a prediction of the graph of the function created by mapping the natural numbers to this sequence, Rayna sketches a concave downward curve starting in the first quadrant and extending into the fourth quadrant. This is the same shape as the last graph she had examined, the graph of the first seven terms of the function created by the geometric sequence -2, -8, -32, -128, . . . , and it is a reasonable prediction for a decreasing function. Using the **Fill** command she fills a new column on the table window from 317 to -300 by having Function Probe subtract 103 from each value to calculate the next value, and six entries are immediately filled in. Leaving the x-axis selector over the first column containing the first seven natural numbers, she then drags the y-axis selector to the new column of sequence values and chooses the **SEND** command which sends this mapping to the graph window. She rescales the y-axis in the graph window from -300 to 320 in units of 100 in order to see all six points sent.

> Interviewer: There it is.
>
> Rayna: Oh!
>
> Interviewer: Why'd you say 'Oh!'?
>
> Rayna: It was just a straight line!

Asked to explain why this sequence becomes a set of points along a straight line and not a curve, she guesses that it is related to the fact that one graph resulted from a geometric sequence and the other from an arithmetic sequence. She is asked why that would affect the graphs, and she says:

> Umm, maybe it's because with geometric you multiply, so your numbers would -- Like you would go from, for example, -2 to -8 to -32. There's so much of a gap between the

numbers. Whereas in arithmetic [sequences] there's gonna be the same amount of gap between the numbers.

She has coordinated an essential feature of an arithmetic sequence, the constant difference between consecutive values, to its graphical counterpart, the constant slope. This concept has not been part of the classroom discussions to this point. Rayna has drawn this conclusion on her own.

Rayna is one of the students who had had algebra II as a junior. Compare the following excerpt of Aubrey, one of five students in the class who had had trigonometry as juniors. In spite of the additional year of mathematics coursework Aubrey does not make the same connections as Rayna. In an interview on February 6, 1990, Aubrey is given the sequence 317, 214, 111, 8, . . . and is asked for the type of sequence and the next term. He subtracts the numbers, determines that it is an arithmetic sequence and finds the next term of -95. Asked to sketch his prediction of the graph created from that sequence, he predicts a concave downward curve starting in the first quadrant and extending into the fourth quadrant. This is the same shape as his previous graph (for the function created from the sequence: -2, -8, -32, -128, . . .) and the same prediction made by Rayna. He uses the **SEND** command to send the table values to the graph window and rescales the graph window.

Aubrey: Hunh. So it's a -- Instead of a curve, it's like a straight line.

Interviewer: Yeah. Why is that?

Aubrey: It's decreasing.

Interviewer: OK. But this one [previous curve] was decreasing.

Aubrey: OK. [pause] It started at positive numbers, whereas this [previous] one started at negative.

Interviewer: OK.

Aubrey: But then it goes into negative numbers.

Interviewer: [The arithmetic sequence] goes from positives to negatives?

Aubrey: Exactly. I think if this one here [the geometric sequence] started at a positive number, it could have possibly went straight also.

Aubrey does not attend to the rate of change of the graphs, and he is unable to draw a conclusion about the relationship between the shape of the graph and the type of sequence used to generate the graph that is based on the defining features of the different types of sequences. He is looking at the quadrants in which the graphs appear but not attending to the rate of change of the values.

Mike is another of the students who did not have trigonometry as a junior. He is quiet in class and not an outspoken member during group work, but his interview protocols show that he is making important connections. Following an examination of the sequence -2, -8, -32, -128, . . . and the graph constructed from it, on February 6, 1990, he is given the sequence 317, 214, 111, 8, ... He enters the values in a table column and finds the ratios between the values using the **Ratio** command. Upon seeing that the ratios are not constant, he then finds the differences between consecutive table values using the **Difference** command. Observing the constant difference of 103 he names the sequence a "decreasing arithmetic sequence" and finds the next term, -95, by subtracting on paper. Asked to predict the corresponding graph he roughly plots the points on paper and extends a line into the fourth quadrant skipping over the x-axis. He explains that he doesn't think there will be a point for the graph on the x-axis, but comparing this prediction to his previous graph of a geometric sequence, he says, "I don't think it will curve off too much." When he then observes the linear graph displayed by sending the table values to the graph window using the **SEND** command, he says, "It doesn't really curve that much . . . it looks like a linear function." Asked to explain why the graph based on the previous geometric sequence, -2, -8, -32, -128, . . ., is curved and why the graph based on this arithmetic sequence, 317, 214, 111, . . ., is a straight line, he replies:

I believe we get the curve on the other one because it's multiplication and when you multiply numbers they get bigger faster. And then when you add numbers or subtract numbers it's always constant. It's always like minus 103, so the slope is actually just a little bit smaller. The slope is smaller here than it is here. I mean it starts out small, the slope on the first one [the geometric

> sequence], and then it goes real big. And then the slope on the one right here [the arithmetic sequence] -- the straight line -- it just remains constant the whole time.

Mike differentiates between the functions by comparing their rates of change. His concept of slope as a rate of change from one point to the next is useful to him in analyzing exponential as well as linear functions. Three days later, in a follow-up interview, Mike is asked to create a geometric sequence that decreases. He opens a Function Probe table, labels the first column 'x' and fills from 1 to 7. In the next column he enters the numbers 20, 10, 5, 2.5, 1.25. In the third column he enters his construction of the equation for the exponential function based on this geometric sequence, $y=20^*.5^{(x-1)}$, and sees that the same values are filled in, thus verifying his equation. Asked to predict the graph of this function, Mike sketches an appropriate shape explaining why the graph levels off at the x-axis:

> It can never go below zero because you're always taking half of the before number -- the number before it. Because it just takes decimal places and decimal places.

Mike has coordinated the graph of this function with the operation of repeatedly multiplying by one half as a decimal. This coordination of operation and graph allows him to make other predictions and develop a classification system all his own.

Mike is then asked to create a decreasing geometric sequence that would appear in the fourth quadrant. He starts by entering $y=-20^*.5^{(x-1)}$ in a new column. However, upon viewing the values which are immediately filled in corresponding to the x values of one through seven, he realizes that this is an increasing sequence rather than a decreasing sequence. Saying, "So instead of taking half of it, I'll multiply by 2," he changes the equation to $y=-20^*2^{(x-1)}$. He predicts a graph that is the same shape as the previous graph (decreasing, concave upward, the first quadrant graph of exponential decay), but located in the fourth quadrant, and he calls the prediction a vertical translation of the previous graph. As he waits for the graph to appear on the screen, he says:

> Come to think of it, I don't think it will level off because it would keep on going down and never get a chance to reach zero. Just go down, down, down. So actually I think it

> might be more of a straighter line. I think maybe it'll come down and it'll keep on going down. It'll never really level off.

"Reaching zero" is Mike's way of describing what may or may not happen at an extreme of the graph of a geometric sequence. Asked why one geometric sequence graph is level and then drops sharply (fourth quadrant, concave downward) and another drops and then is level (first quadrant, concave upward), Mike says:

> Because in the beginning if we start with a negative it's gonna be already up there by zero and it's just gonna keep on going down. But then when we start with a higher number and we decrease, it's gonna go down to zero but not at zero. It's just gonna level off.

To Mike the approach to the x-axis or departure from the x-axis is an important distinguishing characteristic with which to differentiate decreasing exponential functions. Asked to elaborate on why the graphs are different he points to the table window and responds:

> Maybe because here if we would take the negative signs away, these numbers (-20, -40, -80, . . .) would be doubled, and over here (20, 10, 5, 2.5, . . .) it's all half. These ones -- Normally we think of 10 then 5 then 2.5, we think of that decreasing and so it'll be going to zero. And this (-20, -40, -80) is going away from zero. . . . The one with the 10, 5, 2.5, that would be going toward the zero, so it'd be in the first quadrant. And the one with the 20, 40, 80 would be going away from the zero so it'd be going down. That's decreasing.

This analysis is a variation on the rate of change concept. Mike has constructed for himself a way to differentiate between exponential functions that is based on whether the values are getting closer together or farther apart and on whether the points are approaching zero or getting farther from zero. It is a strong and useful construct. It allows him to generalize both the quadrant location of the graph and the shape of the graph.

Discussion

These results provide evidence that the use of a multirepresentational software tool in conjunction with appropriate curriculum and instruction can assist students in constructing viable conceptions of mathematical functions. As other researchers have found (Dreyfus & Eisenberg, 1982, 1984), some students are more comfortable with numerical table representations of functions than with algebraic representations, emphasizing the importance of the availability of multiple representations. By allowing students to build data tables and to be able to see immediately the corresponding graphical representations, especially with graphs which are difficult to plot by hand because of scaling issues, the software provides important feedback for student reflection. Using such a tool, students can examine several graphs in succession or at once if they are superimposed on one set of coordinate axes. By objectifying graphs, multirepresentational software frees students to attend to global features of the graphs of functions such as shape, direction, and location, and thus can lead to stronger conceptions that are based the coordination of actions with representations.

From this small sample of data it is clear that students use multirepresentational software in varying ways. Some interesting uses involve testing guesses or estimates or hypotheses: using the **Ratio** command to verify a conjecture that a sequence is geometric; slowly building an equation and checking the computed equation values against expectations and the values from a **Fill** command; and changing the scale of the graph window to examine the extreme points of a graph. These uses, and many others emerging from the data, point to the important role that multirepresentational software can play in the school curriculum.

Obviously, strong conceptualizations do not depend on computer software alone, but to see a cross-section of students coordinating several representations, reflecting on the relationships between representations, and constructing their own concepts of the rate of change of a function is impressive and encouraging These students have shown that they are able to express functional relationships based on contextual situations in different representations, including algebraic equations, and they are able to develop strong concept images, including the concept of the rate of change, from such tasks. By attacking an interesting problem, by coordinating the actions in the problem with mathematical operations, and by reflecting on the representations which are used to record the operations, students are able to construct highly developed and personally meaningful mathematical conceptualizations.

One of our hypotheses at the outset of this study was that a multirepresentational tool used in a problem-solving-based curriculum would foster students' ability to develop richer conceptualizations of linear functions and of exponential functions. Our analyses of our results indicate that the coordination of multiple representations of functions takes time to develop. As others have stressed (Confrey, 1987; Even, Lappan, and Fitzgerald, 1990; Senk, 1989) the teacher's role is crucial. Teacher-development programs need to address the pedagogical issues of incorporating new curricula and new technologies into the classroom in order to allow students' conceptualizations to develop to their full potential.

Small-group Problem-solving

Group Structures

Although previous research (Davidson, 1989) had indicated that groups of three or four seemed to work best for problem-solving activities, there was little evidence of how this might change when a computer was introduced as the primary tool for representing mathematical ideas, taking actions, and providing a means of communicating solutions. During the summer development session, there was some concern among the teachers that having more than two students per computer might introduce a "goof-off" element, i.e. the opportunity for one or more students not to participate actively. On the other hand, there was concern that groups of two might lead to a dominant/passive relationship. Two possibilities were discussed, three students on one computer or four students on two computers. The second model evolved out of discussion of possible roles for individual students. As a means of keeping a record of group processes and providing a document to encourage reflection, it was suggested that a role of "recorder" be created. The recorder would work on a word-processor on the second computer and be responsible for recording in narrative style the actions and decisions of the group.

Approximately one week before the beginning of classes, the teachers attended a one-day workshop offered by Neil Davidson on small-group work in the classroom. This session focused primarily on general issues of how groups can be chosen, how groups can be assessed, and a series of general exercises that can be used with groups to help them feel comfortable with one another and to foster attitudes of communication. Paula used some of these exercises in the first few days of class to strengthen group interactions.

At the beginning of the school year, Paula decided to use groups of four on two computers. The four roles for group members would be:

1) keyboarder, responsible for using Function Probe;

2) recorder, responsible for recording the thought processes of the group on a word processor;

3) sheepdog, responsible for reading the problem and keeping the group on task;

4) brainstormer, responsible for coming up with new ideas. Roles within a group would be rotated at the completion of each problem. Groups were composed by Paula, and were heterogeneous.[7]

An early change in the model, which occurred within the first three weeks of the school year, was the elimination of the recorder role from the groups. The recorder tended to be involved with his own computer and with the task of recording the events of the group and have little involvement with problem-solving processes. In addition, the recorder tended to concentrate on the products of the group rather than the process. Thus a typical description might include a step-by-step of how Function Probe was used with little discus on of how those decisions were reached. The effect was to produce a document which provided little basis for individuals to relate the development of problematic with group processes and thus had little value for stimulating reflection. Being aware of these issues, after three weeks Paula decided to eliminate the recorder role from the groups and reduce the group size to three.

Group Interactions

It was our intent in initiating this research that we would be able to record

[7]Based on Paula's assessment, each group consisted of one high achiever, one low achiever, and two in the middle.

interactions among members of a small group working in a classroom that could be interpreted as a record of individual students working to define and resolve their own problematic as the group worked toward a negotiated consensus for the problem. To a certain extent this model was based on the teaching experiment (Steffe, 1991; Steffe and Cobb, 1983; Confrey, 1990a) which consists of active questioning of an individual student while working on problems, allowing the interviewer to model the mathematical concepts of the student. We did not intend to interview the students actively, but thought that interactions among students around the computer would provide that role. For a variety of reasons, we did not, for the most part, succeed in capturing such a record. Although technical difficulties (i.e. obtaining an audible audio track of one group working in a classroom with 20 students and six computers) and institutional obstacles (high absenteeism, short class periods, other interruptions) played a significant part, student's lack of practice in talking about mathematics was also important. In undertaking this type of research, one must face the question of why (i.e. under what conditions) it is that students would *want* to talk about solving mathematical problems.

Early in the year, the task for students was strongly oriented toward finding *the* solution to a problem as quickly as possible. Thus group processes tended toward what were perceived as the most efficient means of deriving an answer and little exploration was undertaken. Assigned group roles were often ignored as one, two or sometimes three dominant individuals tended to appropriate the problem-solving process. During this time, Function Probe was often used as a tool for creating the solution. An action would be taken and checked to see if it was a viable solution. If it was, the problem was considered finished. If it was not, it was discarded and another action attempted.

By January, two complementary changes had occurred which tended to promote more exploratory problem-solving processes. First, students had become familiar enough with Function Probe so that trying to decide what the program could do and how to do it was no longer an issue. Second, students had, to some extent, developed a language for Function Probe that allowed for easy communication about what actions might be undertaken. Thus expressions like "fill a column," "make a button," and "move a graph" began to have a meaning for students in the contexts of actions they had previously taken. In retrospect, we realized that students probably need to go through this development process in order to create the possibility of using Function Probe as an exploratory tool and be able to communicate that possibility among members of a group.

In general, however, we did not observe the level of group interactions within the classroom that we had expected, despite a rich context-based curriculum and a teacher committed to valuing student diversity. We believe there are several reasons for this:

1. The first answer is in the form of a question: Why would students want to engage in this type of activity? Given that their educational experience prior to ACOT has not encouraged nor valued this kind of activity, one must assume that creating conditions where it occurs can only happen by actively overcoming resistance. The larger question of the extent to which this can happen within formal educational institutions seems to be open. Schoenfeld (1989) claims to develop a "community of mathematical practice" in his university course, while the work of others (Carraher, 1989; Lave, Murtaugh, and de la Rocha, 1984) may be interpreted as raising the question of whether the context of school is not itself antithetical to this type of learning.

2. We would argue that even if the desire were present, students must learn how to engage in activity at this level. For instance, it was clear that simply instructing the recorder on what he should be doing was insufficient. It must be modelled and practiced, and much may fall under the heading of tacit knowledge. Thus even under the best of circumstances, this is a process, not an event that can simply happen in a classroom.

3. From the teacher's perspective this was an overwhelming intervention. Paula had considered herself a poor mathematician who saw few connection between school math and practical math, who avoided word problems, and who thought that most problems had one best method of solution. After little more than a week's preparation, she was attempting to conduct a context-based curriculum in which she was trying not only to value student diversity, but to evaluate and analyze their diverse methods as she attempted to help them construct new knowledge. In addition, after

using the computer as either a calculator, a tutor, or a means for recording prescribed algorithms, she was now using an open-ended multirepresentational tool and was expected to encourage students to use it in multiple ways. In short, she was overwhelmed by issues of curriculum and technology, and it was unreasonable to expect her to model the group roles constructively in this situation.

4. The forming of a problematic must, to a certain extent, take place within the realm of what one considers possible. By allowing for multiple representations of mathematical ideas and means of carrying out actions, Function Probe can be a medium that allows for the construction of more varied problematics. To be so, however, one must have not only the desire, but also enough familiarity with the tool to imagine what is possible and use it to try out those possibilities. Initially Function Probe was used primarily as a way to make and represent the 'answer.' To use it in creating ideas, students needed first to learn to value diversity, then to value their own ideas, and finally to feel competent in using the tool to represent those ideas. We observed a marked increase in the variety of ways and the frequency with which Function Probe was used as the year progressed, particularly in the second part of the research.

5. The presence of the researchers in the classroom with video cameras and tape recorders almost certainly inhibited the kinds of interactions we sought, especially early in the year. To engage in the type of interactions being asked of them was, for them, to expose their ideas and possibly feel quite vulnerable in a situation in which they had had little voice in determining.

6. Also, the school institution itself was inhibitory in the sense that class periods were short with frequent interruptions and absenteeism was high.

7. Perhaps most important was the need for students to learn to value and learn how to work in the environments that were being created.

<u>Case Studies</u>

In this section, two brief examples will be examined. The intention here is not necessarily to make general claims about the quality of group interactions in the classroom, but to begin to examine what is possible in a classroom. How do we see group interactions between peers as contributing to a construction of knowledge that is qualitatively different from what we would expect to happen when working alone or with a teacher? How can we begin to see these interactions as complementary -- and possibly even essential -- to a constructivist approach to teaching?

Both of these episodes were based on the Parking Garage Problem, given below:

> The mayor of Columbus has proposed four possible rate structures for a downtown parking garage. Students are instructed to derive an equation for each option, using at least one representation from Function Probe.
>
> Option 1: Pay 35¢ for up to, but not including the first hour. Pay an additional 50¢ for up to, but not including, the second hour. Pay an additional 50¢ for up to, but not including, the third hour, and so on.
>
> Option 2: Pay 10¢ for up to, but not including, the first hour. Pay an additional 50¢ for up to, but not including, each additional hour.
>
> Option 3: Pay 35¢ for up to, but not including the first <u>half</u> hour. Pay an additional 25¢ for up to, but not including, each additional <u>half</u> hour.
>
> Option 4: Up to the first hour is free. Pay 75¢ for each additional hour, and so on.

The following excerpt is from three students, Howard, Dan, and Martin as they initially work on the problem. Because of space limitations, some of the confirmatory remarks among the individuals have been left out. In listening to the tape, it is evident that a considerable amount of mutual checking, especially between Howard and Dan occurs during this episode. The analysis on the right is primarily one interpretation of the 'mathematics' of the dialogue. In addition, the discussion is divided into four descriptive episodes of group interaction. These will be discussed further at the end of the example. As the episode begins, Dan has just read the problem through Option 1:

DIALOGUE	ANALYSIS
I. Mutual Rehearsal	
H: All right pay 35 cents.	
D: For the first hour, and for the second hour you have 50 cents if you stay H: But not including the second hour. D: So, if you stay for an hour it's 35.	Dan and Howard begin a process of translation, as they mutually verify the written description of the problem in terms of the actions (addition) needed to implement the description.
H: If you stay for 1 hour and a minute it's 50 cents added there, so it's 85 cents. You see?	
D: You stay for 1 hour it's 35 cents.	
H: So you can stay for 1 hour.	
D: You stay for 1 hour or any amount under 2 hours it's 35 cents plus 50 cents.	
H: You stay for more than 2 hours you're going to pay $1.35.	
D: And for this it's the same thing, but instead of paying 35 you're paying 10 cents for the first hour. The second hour will be 60 cents. The third hour will be $1.10. The third one for every half hour it's 35 cents. And if you stay for an hour it will be 55.	Having apparently reached an agreed "negotiated consensus" of the first option, they continue through Options 2-4, again verifying a fit between action and description in each case
H: You say 60?	
D: The first hour's free, after that it's 75 cents, but not including the second hour. So, the first hour's free, then 75.	

H: One-fifty.

II. Cognitive cooperation

D: Now it's equation time . . ... Got any bright ideas?

H: What, the equation?

M: I'd say, y equals x . . .

Despite the suggestion that students first find a way to represent the problem in Function Probe, the perceived task of 'finding the equation' dominates.

D: No. we're using that one

H: Yeah, remember.

(Several short exclamations about Y)

D: Y equal X plus 35, that wouldn't work.

D: Do you think this would work? OK Y would be the number of hours.

H: Say Y is one.

D: OK. and say X is . . .

H: I think [illegible] should be 35. Then the plus would be an additional 50 cents.

D: See, the 50 cents is constant, but the 35 is . . .

At this point a number of things begin to happen: the first model of an equation is built on a correspondence rather than our more modern notion of equality. The "=" has an interpretation such as "goes with," and the interest is in how one goes from one pair to the next. This is a reasonable way to think of how one would represent the actions described in the initial

H: The 35 is always gonna be there, so that should be X. Yeah, and that should be plus 50. The plus is gonna better it.

D: Plus 50 then.

H: Yeah

D: So, you're saying this is the 35 cents

H: That stays the same.

segment and is also parallel to the way one might Fill the table in Function Probe. Given their initial assumption that the equation is of the form y = x + , it is almost as if Howard is trying to "solve" the equation for x, thus he proposed that x is .35. Howard is proposing a nice model of coordinating actions of time (add one) with the actions of cost (add 50), but cannot find a way to co[illegible] those actio[illegible]

D: Then that should be, why would there be an X in there?

Dan is more concerned with [illegible]ations, questioning why they would use X if it only takes on one value.

[illegible]: If Y is one, X is still going to be 35 cents. We gotta worry about that plus.

D: Well, if Y is two . . . two equals 35, plus 50. [illegible] doesn't equal 35 plus 50.

H: Two represents the number of hours

He is the first to create [illegible] contradiction between the correspondence model and the typical equation model: [illegible] doesn't equal 35 plus 50. [illegible], Howard [illegible]n't bother[illegible] the number of hours and [illegible] [illegible]

III. Cognitive Conflict

D: I think this should equal hours and this should equal something else.

H: Try it. It's not going to work that way. I'm telling you.

D: So, are you saying this should equal the 35 cents?

The ensuing discussion is a period of conflict as Dan seeks alternatives to resolve his conflict while Howard rejects them, maintaining his conviction that the correspondence model is the way to go.

H: Yes! So, 35 cents, you're going to pay it no matter what.

D: Yeah, but then the thing is look, if X is 35 cents why you gonna X there? It's 35 always. So you're saying if we stayed 20 hours, 35 plus 50, that's only 85 cents.

H: The other way it wasn't going to do nothing either, was it?

D: No. That's wrong. That's got to be wrong.

H: 35 right here. We don't know if it's gonna be plus 50 or plus 900. We gotta figure out ....

D: So this can't be the absolute value. Something else has got to be the absolute value, or whatever she was talking about.

M. Because the other way, plus 35 wouldn't work either.

Dan has been observing Paula's description of how to use the greatest integer function on Function Probe (which he mistakenly calls the absolute value) and seems to use this as evidence that success lies in developing this model.

D: So it's something absolute value something plus 35. You know what I think it is? I think it's this, I think it's Y equals the absolute value of X times 50 plus ... something like that

M: What? ... Can't be times

H: Times? ... that's gonna give some high numbers.

D: I know.

It's difficult to see exactly how Dan has derived this conjecture. It is possible that he overheard other students talking, saw the form of the equation on the board, or recalled the form of the equation from previous problems. In any case, he has continually been more tuned into the formal mathematics than Howard and now offers a conjecture ("times 50") that is at first seen as unbelievable by both Mark and Howard.

H: Times 50 (chuckles).

D: (talks to self: but ... point 50, ... point 35)

Dan continues to work on the equation, suggesting apparently to himself that if 50 is too big, perhaps .50.

IV. Negotiating Consensus

H: It might work. Cause if Y is one ...

D: We wouldn't even need a one

H: Something's got to be one, because if it's two hours, then -- if it's two hours it's gonna go up to $1, and plus the 35 is $1.35. So, it is times 50.

Howard, more the pragmatist, is now playing with Dan's conjecture, deciding that it might work after all and is in fact now unconcerned with Dan's offer that perhaps .50 will work.

D: ...es point 50.

H: Yeah, so we got Y equals ...

The end.

Unfortunately, Dan did not come to school the following day, and Howard decided not to work alone, thus we do not know how they might have proceeded. It appears that both Dan and Howard are accepting the action of 'times 50' (or .50) as viable for the problem and one can only speculate on how they may have continued in deriving an equation. However, it is noteworthy that the two competing approaches (multiply by 50; add 50) were both reasonable ways to model the problem. The preference for the former was simply based on the norms for the task of writing an equation. If the task had been to create a table representation, the latter approach (add 50) may have been preferred.[8]

Several questions arise from this example: whether the exploration of competing approaches and the changes that result from going through successive stages (in this example mutual rehearsal, cognitive cooperation, cognitive conflict, and negotiation of consensus), are less likely to occur for students working alone or with a teacher; and how knowledge constructed may be different as a result of these kinds of collaborative processes. We tentatively suggest that there is evidence that the collaborative process may be qualitatively different, and we intend to explore these questions further as we examine more examples from our research and do more work of this nature.

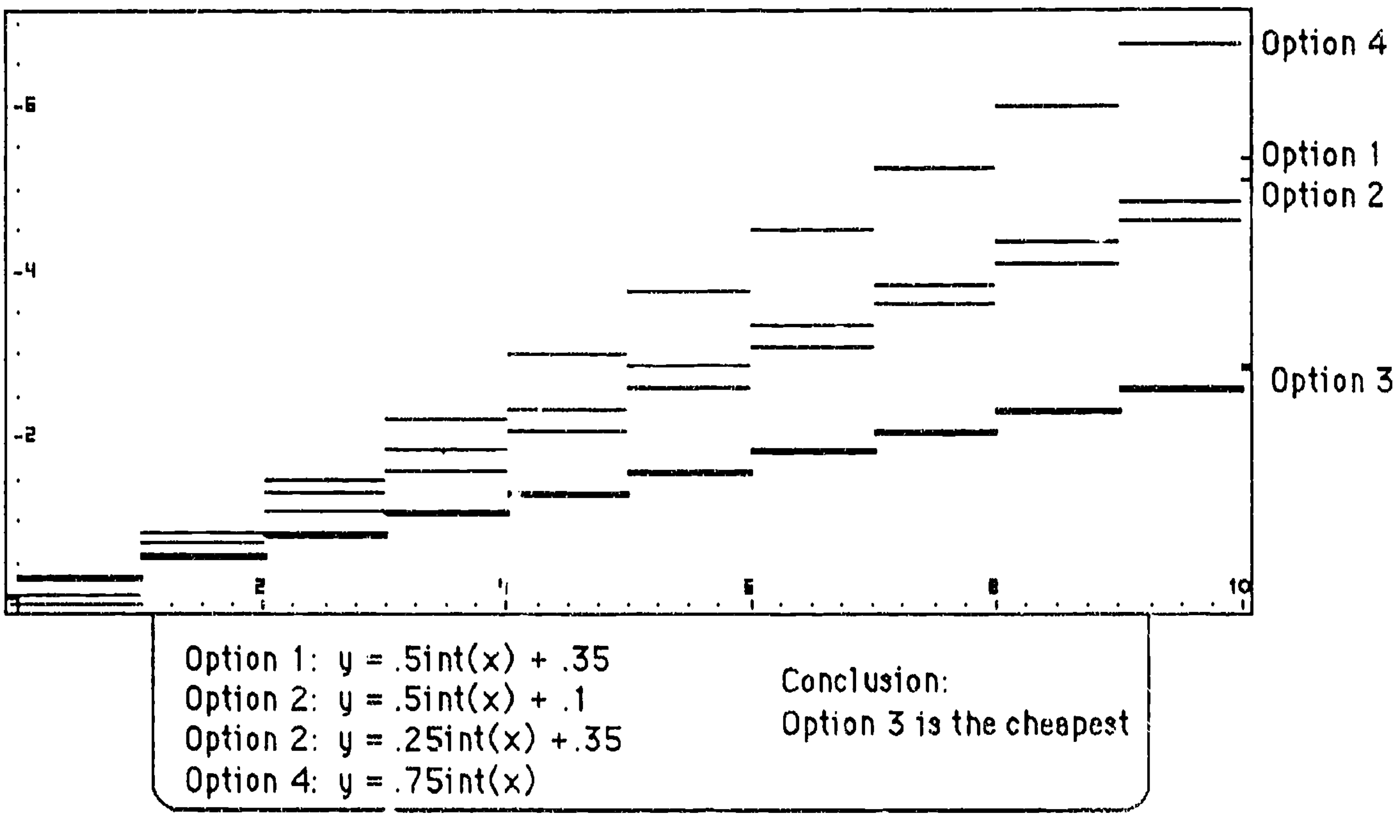
Group presentations tended to be a source of lively and rich student interactions. One can look at these whole-class episodes as similar to group work in that knowledge is constructed and consensus is negotiated through a process of social exchange. There is also a difference. Although students had no externally compelling reason to join the discussions, they also had nothing to lose, as they were not involved in a perceived task for which discussion was seen as a diversion. Also alternative approaches could be offered and defended from the position of having previously reached a negotiated consensus within one's group.

The example which follows occurred two days after the above episode. Howard has not seriously worked on the problem since the day with Dan, and thus has not worked out a satisfactory equation. Although Howard is often outspoken and a

---

[8] In general, we write our curriculum with an intention of encouraging students to use several representations, so that they begin to understand how actions can be represented differently. Although using another representation was suggested in this problem, Dan and Howard apparently viewed the task primarily as getting the equation.

leader in class, a researcher on our project who interviewed him described him as shy and very weak on formal mathematics, particularly in deriving equations. As the above episode shows, however, he appears to have strong intuitions and is pragmatic in making sense of mathematics. We will see these qualities evidenced again as he participates in a class interaction during a group presentation in which the formal mathematics of two students, generally perceived by their peers to be strong math students, comes under scrutiny.



As described in the section on "Students" above, Chip and Katrina derived the equations for the parking garage problem quickly and showed strong convictions in their belief in them. Each was able to articulate concepts of intercept as initial value and multiplier as the hourly rate of change. The next episode comes from their class presentation. A facsimile of the overhead they used in the presentation is shown above.

The episode begins as Katrina has finished explaining their solution to the class and declared Option 3 as the cheapest and thus the one that the city should adopt. Katrina and Chip have neglected to take into account the half hourly changes in Option 3. Thus the equation should be: y=.25int(2x)+.35. It is not the cheapest option.

DIALOGUE

H: You say Option 3 is cheaper. How is it cheaper?

K:: If you look at your lines, the x-axis is the hours past and the y-axis is the amount you pay. (Pointing at graph) No matter, well it's not cheaper for the first hour, but after the first hour its the cheapest one there is.

H: So it's cheaper after the first hour. OK.

K: If you go to the City Center, you'll stay more than an hour.

H: If you're going to be there for longer than an hour, it's cheaper.

(A few minutes later)

H: Question: how are we supposed to read this like every step. Like at five hours there's two points in there. Which one am I supposed to read?

C: Read the upper value.

H: Yeah.

A: Your equation on Option 3. Your graph goes to the first half hour at 35.

C: It's at 35 all the way up to the first hour.

ANALYSIS

Although Howard has not derived any formalized solutions to this problem (he has not worked on it since the above episode), he did develop a strong intuitive sense of the problem. One can sense him testing his understanding of the problem against what he considers stronger mathematics students and formalized mathematics. However, it's not until Aubrey seems also to be questioning the solution that Howard feels more confident in his challenge.

The challenge is issued.

S1: It's supposed to be .25.

H: Yeah, it's supposed to be .35 plus .25.

S2: That equation doesn't work because we tried it.

H: Can I say something real quick? The first half is 35. When does it start adding? What about the 25?

K: What do you mean?

Although Howard did not himself derive an equation, he did develop an alternative approach which he is now able to offer in the context of the conflict.

H: The first half hour is 35. What about the 25¢. Is that excluded? That's not shown in the formula.

K: The third option is 35¢ for the first hour, so the first hour would be . . .

C: (Quietly) Half hour . . .

K: First half hour would be 35¢.

C: (Pointing at graph) As you can see on here, at .5 it increases, or at .5 you can see it. (Points to 0.5 on x-axis). It jumps up (here).

(Approx. 10 seconds of murmuring as students comment to each other)

Chip and Katrina are confident of their solution, and seem surprised at the building challenge. Even after Chip realizes that it should change every half hour, he is not ready to give up on his solution, seeming to believe that his graph must jump every half hour. They are very reluctant to reexamine their solution, having been so confident the previous day in explaining their methods to a member of the research team.

C: So it goes across to .5 then jumps up.

A: What I'm saying is does this x (points to Option 1) and this x (Option 2) and this x (Option 4) all represent an hour?

Aubrey, who often plays the dervish role is happy to lead Chip into a trap.

K & C: Yeah.

A: And then you're changing this x to a half hour?

C: Right.

A: But they're all supposed to correspond together.

C: It's still an hour, it should be .5x inside.

(Pause)

Chip is now ready to reexamine his equation and quickly offers an amended version, .5x inside.

A: (triumphantly) You're equation won't work.

H: If you put .25 in for x and times it by .5, you're not going to get .6 for the first half hour.

T: I can barely keep up with you. Why do you keep concentrating on .5 for x. Why don't you use .6 for x?

Although it seems like C, K, H, and A are nearly ready to create an equation that will work for Option 3, Paula, who has misinterpreted the last statements by C and H preempts the discussion with a rote exercise on simply evaluating the equation for various values of x. At this time other questions arise and the period ends. This issue is not resolved until the next day when another group, possibly as a result of this discussion, presents a correct solution for Option 3.

Although it would be easy to be critical of Paula for interrupting what appeared to be a productive conversation possibly nearing a resolution, it must be remembered that Paula is going through this fairly challenging material for the first time and the four students involved had quite different approaches to the problem. It is not possible to know what other factors in the classroom affected her decision to interrupt. However, it also points out the difficulty a teacher can have in promoting group or class discussion in which students take ownership when the teacher has not herself had the opportunity to explore the richness of the subject matter.

In effect, we see episodes such as the two above as providing examples of what can happen in classrooms when students are encouraged to explore mathematical ideas collaboratively, although for such episodes to occur regularly and frequently can only happen over time and with effort on the part of the teacher and the institution. We suggest that the knowledge being constructed through such interactions would likely not occur from individuals working alone or with teachers (or other 'more knowledgeable others').

Discussion

The research on small group interactions yielded three major results:
1) collaborative group work can play a critical role in the development of diverse approaches and solutions to problems, and it assumes an important part in students' developing conceptions of the problem-solving process; 2) effective collaborative learning cannot be implemented rapidly but must grow in conjunction with other classroom practices, including changing the character and duration of teacher/students interactions, assessment practices, and ways of reporting on the outcomes of the group interactions; and 3) although we found that in centering problem-solving activity around a computer, groups of four were too large, this was primarily due to physical limitations rather than the computer becoming a de facto fourth group member. When the computer plays the role of tool, rather than tutor, it can potentially develop a unique communicative function within group activity, promoting exploration in a mutually understood framework of actions and representations.

Earlier, we described three theoretical constructs which we considered essential in understanding how successful collaborative learning might occur. Of these,

Activity is, in a sense, the larger picture in which education takes place, the frame in which Confrey's cycle of problematic-action-reflection cycle might occur. In each case, these models are built by researchers, but to be successful they must be, in a sense coopted by the learner. Thus, just as Confrey has proposed that successful individual learners will, perhaps unconsciously, adopt her cyclic process as their understanding of what it means to learn, we also suggest that successful collaborative learners must begin to view their educational activity in terms of the interweaving of individual problematic and group development of negotiated consensus. As described earlier, activity in this sense and institutional goals that promote efficiency in accomplishing tasks may be in continual conflict.

Recognizing the importance of the individual problematic in collaborative learning is essential. As Roschelle and Behrend (1990) have pointed out, it is in the continual attempt to share a sense of problem that collaborative learning occurs. It is not a shared problem (or problematic) but a continuous attempt to share that creates the opportunity for learning. Thus, we suggest that the different approaches taken by Dan and Howard in their attempts to find equations for the Parking Garage Problem could have led to stronger results (for them) than the more uniform and formal approach of Chip and Katrina.

Finally the group construction of negotiated consensus -- by which we mean the carrying out of actions and creation of representations that mutually resolve the problematics of individual group members -- is essential not only for allowing the group to function but in creating ways in which individuals learn to act, communicate, and represent their very personal ideas in ways that allow them to be productive members of society.

There is considerable evidence that substantial progress toward meeting these objectives was made in the ACOT classroom this year. In particular:

1. Paula has been actively experimenting with alternative forms of assessment in an attempt to balance goals of group responsibility with the school's need for individual accountability. She seems committed to the idea that collaboration is a valuable part of learning mathematics.

2. As Paula's subject matter knowledge grows, her ability to encourage individual expression and processes of student interaction will increase. Given her strong ability to hear the student voice, she could become an exemplary teacher if provided sufficient support and encouragement in these directions.

3. The active class discussions that often occurred around group presentations provided a basis, we believe, for students to value both diversity and their own opinions. Although less formal group work in the classroom occurred during the last half of the school year, students seemed more

willing and occasionally insistent on offering an alternative approach to problems discussed in class.

4. An attitude of openness developed in the classroom that allowed students to work freely with one another on problems as well as to express differences.

5. Although the use of Function Probe was limited in the early part of the year, its spontaneous use by students to explore ideas grew tremendously over the year. As an open-ended multirepresentational tool, built on socially accepted forms of action and representation, it can provide an ideal site for students to take actions individually that others can use not only by interpreting them in the process of negotiating consensus, but also by using the results from their peers to take their own actions.

Further Research

Based on the results of this research, we have undertaken an additional research project looking at small-group activity in a computer-centered mathematics classroom in Ithaca. Although this research continues to focus on the concepts of activity, problematic, and negotiated consensus in group problem-solving, we have used the results of this project to extend and enrich our model. We are particularly interested in two further and related aspects. Since Function Probe can potentially play the role of an action-based means of communication, we believe it may provide a medium that encourages active '"appropriation" (Newman, Griffin, and Cole, 1989). That is, acts of communication using Function Probe, by combining both action and representation allow a potentially richer basis for students to appropriate the actions and suggestions of others. This combination of action and representation creates a basis for appropriation to occur in relation to the student's individual problematic. In addition, when one member of a group creates a representation of a problem situation in Function Probe, the question of *how* that representation was created need not be discussed, since all members are presumably familiar with the actions by which Function Probe creates these representations. Thus a reasonable question to ask may be whether group discussions revolve more around questions of *why* than *how*. Preliminary results from this research suggest that this may be the case.

VIII. CONCLUSIONS

The project, conducted at the Apple Classrooms of Tomorrow with the support of Apple Computer, Inc and the National Science Foundation, provided a stimulating opportunity to consider both the possibilities and the hurdles of creating a vision for the role of technology in teaching mathematics.

From this project work, we would suggest that:

1. If we wish to inspire, test, and revise new technologies that will meet the needs of the educational system, there must be support for research and development at actual school sites. These new technologies should include, in addition to hardware and software tools, materials for use with students in the form of curricula and innovative teacher development materials. Creating change in the public school educational system requires at least as much attention to creating and maintaining innovation within recalcitrant institutions as to producing the good ideas themselves. Innovations that do not take into account and respond to the constraints of state-mandated testing, oppressive teaching loads, and inadequate preparation of teachers by teacher education institutions will not endure.

2. This is not to say that change must be incremental and conservative. The need to change the educational system is extreme. Students are being turned off to mathematics while never having had the opportunity to explore its pleasures nor to develop their own sense of its applicability to their lives. The system that we seek to change is self-reinforcing in its preparation and expectations of teachers, its forms of assessment and evaluation, and its overriding lack of imagination and challenge. Change must be systematic, must be embraced and invested in by the change agents -- the teachers themselves -- and must make taking risks to try new methods worthwhile and effective. This can be done only with the cooperation and commitment of private corporations, government and state institutions, and colleges and universities working together with public school personnel. Demonstration sites such as ACOT are essential to such a process.

3. When creating change in technological areas, we need to be certain that the educational goals drive the innovations. For example, we have witnessed teachers being encouraged to make use of technology in ways that deflect their students' attention from the subject matter. In one case, students took textbook information and simply transferred it, without critical examination or significant revision, to a flashy technological presentation. It is not difficult to see how this can happen given the rapid changes in technology, the pressures put upon schools to adopt new technology at any cost, and the difficulty in producing the desired outcomes when working with materials with which we have so little experience. This use of technology in schools will undermine the importance of schooling for critical thought. Thus, we recommend that subject matter and/or pedagogical goals determine the appropriateness of introducing new technology and not vice-versa. Technology should be brought into the educational process as a tool, and not as an end in itself.

4. The warning about avoiding the seduction of the newest technology while neglecting the curricular goals should not be taken as advocating for a conservative use of technology. A well-designed technological innovation can transform the subject matter in both predictable and unpredictable ways. Research on how technology changes one's understanding of mathematics is at least as important as research on how it allows one to learn the traditional mathematical topics more effectively and insightfully. In our work, the use of Function Probe has allowed us to examine in a qualitatively different way how functions can be tied closely to contextual applications and how they can be approached through multiple representations. In addition, examining the students' methods while using Function Probe, we have found new ways to think about many of our mathematical conceptions.

5. At the same time that we need to produce software that anticipates the dramatic advances in technological innovation, we need to be careful to ensure that issues concerning the access to technological resources do not exacerbate the problems of access to educational opportunity. In mathematics, this requires us to pay careful attention to providing innovations that will reach both women and minorities and encourage them to enter and persist in the mathematical fields.

These are the underrepresented groups we need to seek out and for whom we need to make mathematics more accessible.

6. Software development, especially when it includes research, revision, and the preparation of materials, is expensive and difficult to do within the constraints of a university setting. Keeping good programmers, getting access to new technology, having software count towards academic progress, and finding time to balance design and evaluation efforts can be trials when working in a university setting. However, university faculty in education have certain advantages in that they can have close relationships with schools, access to most recent research results, and freedom to innovate. As a result, it is important to encourage cooperation among the university, corporations, government, and publishers if software is going to have a significant impact in educational settings.

7. There is a key role for teacher development in this process. Function Probe was successful, because although it appears to be closely related to the curriculum mandated in the schools, it can lead to radical changes in how students and teachers approach problems. We have found that it allows teachers to develop a deeper understanding of subject matter, not by reteaching material that they have experienced, but by approaching that material in a different way. There is a strong need to provide teachers with opportunities to explore subject matter in depth using a variety of approaches. We believe Function Probe can play a significant role in providing an opportunity for such exploration.

In conclusion, our experience has convinced us that technology can be integral in the revitalization of mathematics classrooms across the country. That process needs to be informed by a clear image of what mathematics teaching and learning can be like and a principled use of technology to achieve those ends. The necessary products will include tool-based software, interesting curricular innovation (perhaps in the form of multimedia environments), testing and assessment measures, and teacher-development materials. However, the process will begin in earnest only as individuals create compelling examples of such software tools and of the ways they can be used productively in classroom settings. It is our hope that we have contributed in some modest way to such an effort.

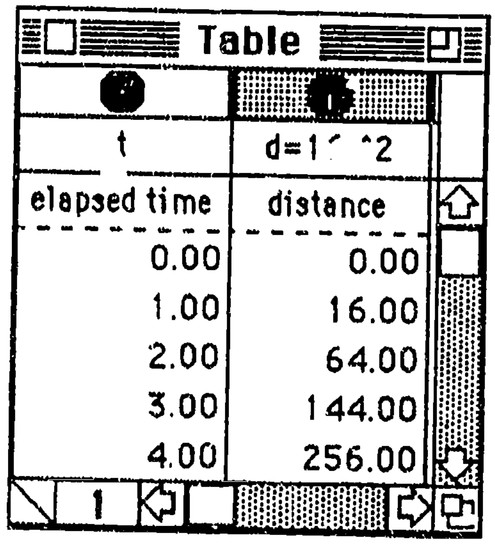
APPENDIX – A BRIEF DESCRIPTION OF FUNCTION PROBE

Function Probe© is a multi-representational tool for learning about functions. By calling it a "tool", we wish to indicate that the program is intended to be actively used by students in their problem solving efforts. Thus Function Probe is neither a tutorial nor tied to specific curricular materials, but instead designed to be compatible with actions and representations which students create and use in a variety of problematical situations which they may encounter while studying mathematics.

The tool currently has three windows: a graphing window, a calculator, and a table-making window. Each window is designed to encourage the learner to operate actively in ways compatible with that representation. In addition, students can pass functions and data among the representations allowing them to develop flexibility and insight in their understanding of the concepts. For the convenience of both student and teacher, a history is kept in each window of the actions the learner has undertaken.

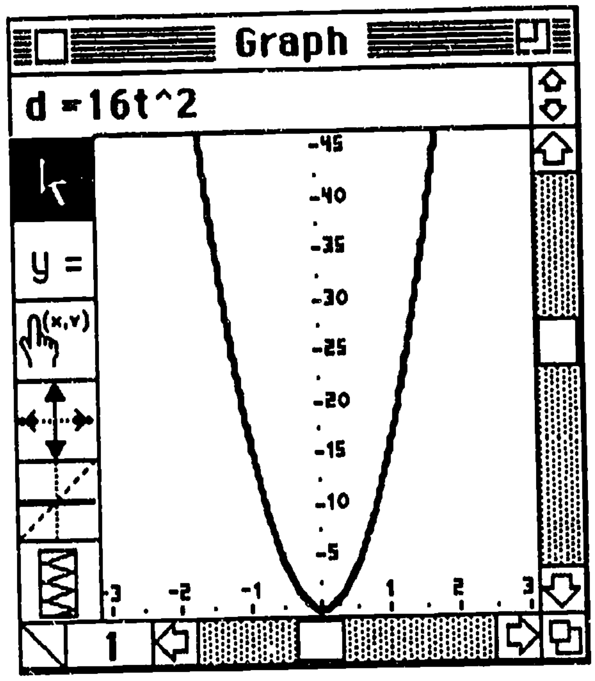
A description of the features of each window follows:

The Table Window



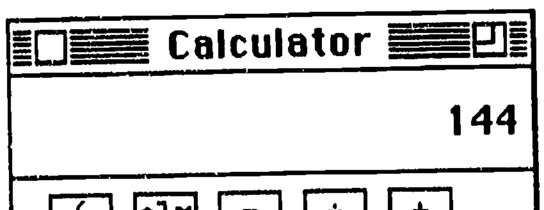
The table window is similar to a spreadsheet with the important exception that functional relationships are between columns rather than individual cells. The software allows for entry of data values individually, or by iteratively filling a column through addition, subtraction, multiplication, or division. Thus, many functional relationships can be represented by filling columns, rather than requiring equations. Columns can be named with both formal and informal labels. Formal labels must be single variables which can be used to build up functional relations. Informal labels can be extended as needed. Learners can sort columns, and /examine differences and ratios of successive values.

The Graph Window



Either discrete points or continuous equations may be entered in the graph window. In both cases the functional relationship is maintained allowing the point sets/equations to be translated, stretched and/or reflected as functions. A register is designed to keep track of the magnitude and direction of the transformations and this information is stored in the history window. A pointer icon can be used to see the coordinates of points or to input new points. Multiple graphs can be graphed and selected independently. Functions can also be inputted algebraically or imported from other windows.

The Calculator Window



In addition to having the functionality of a basic scientific calculator, the calculator window is designed to encourage the view of functions as generalizations on procedures. Procedures, represented by calculator keystrokes, can be made into a "button" by selecting the desired string from the keystroke history. These buttons can then be called in the calculator by a single keystroke, or "imported" to the graph or table window. We have found students to be adept at representing mathematical relationships through keystroke records and quick to use buttons as a means of preserving these relationships for future use.

Recent Changes and Plans for the Future

Since the completion of this research project, several new features have been added to Function Probe and more are in progress.

Table Window

The table window now allows user-selected columns to be linked for sorting purposes. The **Fill** command allows the specification of a number of iterations and allows the user to enter either a single operation (+, -, *, ÷) or an algebraic expression for iterating succeeding values. In addition to the **Difference** and **Ratio** features, choosing **Accumulate** creates a column showing the successive accumulation of the values in a given column. Columns can now be moved, inserted and/or deleted. A means for importing data from other sources is currently under design.

Graph Window

The graph window now includes a sketching tool allowing the user to freely draw figures on the graph. From these drawings, discrete points at user-defined intervals can be captured and sent to the Table window. A grid for the graph is currently under design.

Algebra Window

When completed, the algebra window will allow for entry and manipulation of algebraic expressions on the computer. A palette at the side will be used to transform expressions into parentheticals, square roots, exponentials, numerators, and/or denominators. The computer will be able to verify the equivalence of two expressions. Functions can be defined in the window and sent to any other window. Finally the algebra will have a formula-building capacity by which a learner can generate a formula and have the computer query him/her on the desired values for the parameters. These formulae can be stored for future use.

Requirements and Availability

Although Function Probe will currently run on Macintosh computers with one megabyte of memory, because of the size and complexity of the program, we recommend that it be used on computers with at least two megabytes of memory Future versions of the program may require two megabytes of memory. At this time (May, 1991), Function Probe is not available for public distribution. However, we are very interested in seeing it used in both classrooms and as a research tool. If you are interested in getting a copy of the program, or if you have comments or suggestions, we strongly encourage you to contact us.

Dr. Jere Confrey
Department of Education
Kennedy Hall
Cornell University
Ithaca, NY 14853
607-255-1255
FAX: 607-255-7905

BIBLIOGRAPHY

Afamasaga-Fuata'i, Karoline (1991). Students' Strategies for Solving Contextual Problems on Quadratic Functions. Unpublished doctoral dissertation, Cornell University.

Bruner, J. S. (1960). The Process of Education. Cambridge, MA, Harvard University Press.

Carraher, Terezinha (1989). Negotiating the results of mathematical computations. International Journal of Educational Research, 13 (6), (pp. 637-646).

Champagne, A. B., Gunstone, R. F., & Klopfer, L. E. (1983). Naive knowledge and science learning. Research in Science and Technology Education, 1 (2), (pp. 173-183).

Cobb, Paul, & Steffe, Leslie (1983). The constructivist researcher as teacher and model builder. Journal for Research in Mathematics Education, 14 (2), (pp. 83-94).

Cole, Michael. (1985). The zone of proximal development: where culture and cognition create each other. In J. V. Wertsch (ed.), Culture, Communication, and Cognition: Vygotskian Perspectives, Cambridge Cambridge University Press. (pp. 146-161).

Confrey, Jere. (1991a, to be published). Learning to listen: a student's understanding of powers of ten. In Ernst von Glasersfeld (ed.), Constructivism in Mathematics Education, Dordrecht, Netherlands Kluwer Academic Press. (pp. 141-178).

Confrey, Jere. (1991b, to be published). Splitting, similarity, and rate of change: New approaches to multiplication and exponential functions. In G. Harel and J. Confrey (eds.), The Development of Multiplicative Reasoning in the Learning of Mathematics, Albany, NY SUNY Press.

Confrey, Jere (1991c). The use of multiple representations in the teaching of quadratic functions. A paper presented at the Workshop for Ithaca High School Mathematics Teachers. Ithaca, NY, February 27, 1991.

Confrey, Jere. (1990a). The concept of exponential functions: a student's perspective. In Les Steffe (ed.), Epistemological Foundations of Mathematical Experience, New York Springer-Verlag. (pp. 124-159).

Confrey, Jere. (1990b). A review of the research on student conceptions in mathematics, science and programming. In Courtney Cazden (ed.), Review of Research in Education, 16 , American Educational Research Association. (pp. 3-56).

Confrey, Jere. (1989). Student conceptions, representations and the design of software for mathematics education. Proceedings of the Apple's Designing for Learning Conference, Cupertino, CA, July 17-18, 1989.

Confrey, Jere (1987) An epistemology of practice and its relationship to research. A paper presented at the Annual Meeting of the American Educational Research Association. San Francisco, April, 1986.

Confrey, Jere and Smith, Erick. (1988). Student centered design for educational software. Proceedings of the First Annual Conference on Technology in Collegiate Mathematics, Columbus, OH, Oct. 27-29, 1988. Addison-Wesley.

Davidson, N. (1989). The small-group discovery method in secondary- and college-level mathematics. In Neil Davidson (ed.), Cooperative Learning in Mathematics, New York, Addison-Wesley.

Davidson, N. (1989). Introduction and overview. In Neil Davidson (ed.), Cooperative Learning in Mathematics, New York, Addison-Wesley. (pp. 1-20).

Dreyfus, T., & Eisenburg, T. (1984). Intuitions of Functions. Journal of Experimental Education, 52 (2), (pp. 77-85).

Dreyfus, T., & Eisenberg, T. (1982). Intuitive Functional Concepts: A Baseline Study on Intuitions. Journal for Research in Mathematics Education, 13 (5), (pp. 360-380).

Even, R.; Lappan, G. & Fitzgerald, W. (1990). Preservice teachers' conceptions of the relationship between functions and equations. In M. Behr; C. Lacampagne and M. Wheeler (eds.), Proceedings of the Tenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, 21 Dekalb, IL, November, 1988. (pp. 283-289).

Eylon, B., & Linn, M. (1988). Learning and instruction: An examination of four research perspectives in science education. Review of Educational Research, 58 (3), (pp. 251-301).

Feurzig, W. (1988). Apprentice tools: students as practitioners. A paper presented at the Tenth Annual Meeting of PME-NA. October, 1988.

Goldenberg, Paul. (1988). Mathematics, metaphors, and human factors. In P. Goldenberg; W. Harvey; P. Lewis; R. Umiker; J. West and P. Zodhiates, Mathematical, technical, and pedagogical challenges in the graphical representation of functions, Newton, MA Education Development Center.

Greeno, James G. (1988). The Situated Activities of Learning and Knowing Mathematics. In M. J. Behr; C. B. Lacampagne and M. M. Wheeler, North American Chapter of the International Group for the Psychology of Mathematics Education., 10th Annual Mtg. DeKalb, Illinois, November 3-5, 1988. (pp. pg. 481-521).

Herscovics, Nicolas. (1982). Problems related to the understanding of functions. In G. van Barneveld and H. Krabbendam Conference on Functions, Enschede, Netherlands, 1982. (pp. 67-84).

Kaput, J. (1988). Applying technology in mathematics classrooms: it's time to get serious, time to define our own technological destiny. A paper presented at the AERA symposium: Classroom Learning and the Challenge of Technology. New Orleans, April, 1988.

Kaput, J. (1986). Information technology and mathematics: opening new representational windows. Journal of Mathematical Behavior, 5, (pp. 187-207).

Lakatos, Imre (1976). Proofs and Refutations: The Logic of Mathematical Discovery. In J. Worrall and E. Zahar (eds.), Cambridge Cambridge University Press.

Lampert, Magdalene (1988). Teachers' thinking about students' thinking about geometry: the effect of new teaching tools. Technical report #TR88-1, Cambridge, MA Educational Technology Center, Harvard Graduate Center of Education.

Lange, Jan de (1987). Mathematics, Insight and Meaning. Ultrecht, Netherlands, OW & OC.

Lave, J., Murtaugh, M., and de la Rocha, O. (1984). The dialectic of arithmetic in grocery shopping. In Barbara Rogoff and Jean Lave (eds.), Everyday Cognition, Cambridge, MA Harvard University Press. (pp. 67-94).

Leontyev, A. N. (1981). Problems of the Development of the Mind. Moscow, Progress Publishers.

Mandinach, Ellen (1989). Using Stella modeling software in mathematics instruction. A paper presented at the Computer Integration in Precollege Mathematics Education Conference. Stevens Institute of Technology, Hoboken, NJ, April 26, 1989.

Markovits, Z., Eylon, B., and Bruckheimer, M. (1983). Functions - linearity unconstrained. Proceedings of the Seventh International Conference for the Psychology of Mathematics Education, Israel, 1983.

Meira, Luciano L. (1991). Developing Mathematics Sense-making Through Manipulation of Physical Devices. Unpublished doctoral dissertation, Univ. of Calif. at Berkeley, Berkeley, CA.

Monk, Steven (1989). Student understanding of function as a foundation for calculus curriculum development. A paper presented at the Annual Meeting of the American Educational Research Association. San Francisco, March 27-31, 1989.

NCTM (1989). Curriculum and Evaluation Standards for School Mathematics. Reston, VA, Nat'l. Council of Teachers of Mathematics, Inc.

Nemirovsky, Ricardo (1991, to be published). Notes about the relationship between the history and the constructive learning of calculus. Proceedings of the Segundo Simposia Internacional sobre Investigacion en Educacion Matematica, Cuernavaca, Mexico, 1990.

Nemirovsky, Ricardo and Rubin, Andee (1991). "It makes sense if you think about how graphs work, but in reality ... ." Technical Education Research Center, Unpublished paper.

Newman, Dennis, Griffin, Peg, & Cole, Michael (1989). The Construction Zone: Working for Cognitive Change in Schools. Cambridge, Cambridge University Press.

Osborne, R. J., & Wittrock, M. C. (1983). Learning science: A generative process. Science Education, 67 (4), (pp. 498-508).

Pea, R. (1987). Cognitive technologies for mathematics education. In A. Schoenfeld (ed.), Cognitive science and mathematics education., Hillsdale, NJ, Erlbaum.

Rizzuti, Jan and Confrey, Jere (1988). A construction of the concept of exponential functions. Proceedings of the Tenth Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA), Dekalb, Ill.

Rosch, Eleanor (1977). Universal and cultural specifics in human categorization. Empirical Studies in Perception and Cognition,

Roschelle, Jeremy and Behrend, Stephanie (to be published). The construction of shared knowledge in collaborative problem solving. In C. O'Malley (ed.), Computer Supported Collaborative Learning,

Roseberry, A. S., & Rubin, A. (1989). Reasoning under uncertainty: Developing statistical reasoning. Journal of Mathematical Behavior, 8, (pp. 205-219).

Schoenfeld, Alan (1989). Ideas in the air: speculations on small group learning, environmental and cultural influences on cognition, and epistemology. International Journal of Educational Research, 13 (1), (pp. 71-88).

Schoenfeld, Alan H.; Smith, John P. , III and Arcavi, Abraham. (1989). Learning: the microgenetic analysis of one student's evolving understanding of a complex subject matter domain. In R. Glaser (ed.), Running Heading: Learning, Advances in Instructional Psychology, 4 Hillsdale, NJ, Erlbaum.

Senk, S. (1989). Toward school algebra in the year 2000. In S. Wagner and C. Kieran (eds.), Research Issues in the Learning and Teaching of Algebra, Reston, VA, NCTM/ LEA.

Steffe, L. (1991, to be published). The constructivist teaching experiment: illustrations and implications. In E. von Glasersfeld (ed.), Radical Constructivism in Mathematics Education, Dordrecht, Netherlands, Kluwer Academic Press.

Steffe, L. and Cobb, P. (1983). Multiplicative and divisional schemes. Proceedings of PME-NA, Montreal, 1983, (pp. 11-27).

Steffe, L., Glasersfeld, E. von, Richards, J., & Cobb, P. (1983). Children's Counting Types: Philosophy, Theory, and Application. New York, Praeger.

Treffers, Adrian (1987). Three Dimensions. Dordrecht, Netherlands, Reidel.

van Barneveld, G. and Krabbendam, H. (eds.) (1982). Conference on Functions, Enschede, Netherlands, 1982.

Vergnaud, Gerard (1988). Multiplicative structures. In M. Behr & J. Hiebert (eds.), Number Concepts and Operations in the Middle Grades, Reston, VA, National Council of Teachers of Mathematics.

Vinner, S. (1983). Concept definition, concept image, and the notion of function. Journal for Research in Mathematics Education, 14 (3), (pp. 293-305).

von Glasersfeld, E. (1990). An exposition of constructivism: why some like it radical. Journal for Research in Mathematics Education, 4, (pp. 19-31).

Wiske, M., Zodhiates, P., Wilson, B., Gordon, M., Harvey, W., Krensky, L., Lord, B., Watt, M., and Williams, K. (1988). How technology affects teaching. Technical Report 87-10, Cambridge, MA, Educational Technology Center, Harvard Graduate School of Education.

Yackel, Erna, Cobb, Paul, and Wood, Terry (1990). Learning through interaction: a case study. A paper presented at the American Educational Research Association. Boston, April 16-20, 1990.